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AXIALLY SYMMETRIC PLASTIC DEFORMATIONS IN SOILS

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A theoretical investigation is given of quasi-static axially symmetric plastic deformations in soils. The mechanical behaviour of a natural soil is approximated by that of an ideal soil which obeys Coulomb's yield criterion and associated flow rule, with restriction to rigid, perfectly plastic deformations.

There are considerable variations in the structure of the associated stress and velocity field equations for the various plastic régimes, but it is noteworthy that real families of characteristics occur in all non-trivial cases. Attention is focused on those plastic régimes agreeing with the heuristic hypothesis of Haar & von Kármán as being seemingly of application to certain classes of problems, in particular to those of indentation. The stress and velocity fields are then hyperbolic with identical families of characteristics, and the stress field is statically determinate under appropriate boundary conditions.

In applications of the theoretical analysis, attention is confined to situations involving only the Haar & von Kármán plastic régimes. First, possible velocity fields are obtained for the incipient plastic flow of a right circular cylindrical sample of soil subjected to uni-axial compressive stress parallel to its axis. Secondly, a complete solution is obtained for the incipient plastic flow in a semi-infinite region of soil, bounded by a plane surface, due to load applied through a flat-ended, smooth, rigid, circular cylinder; numerical results obtained for this problem include the variation of yield-point load with angle of internal friction of the ideal soil. These applications relate to problems of the mechanical testing of soil samples and of load-bearing capacity in foundation engineering.

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NOTATION

A short list of symbols is given below. All other notation is defined when first introduced in the paper. It should be noted that a few symbols have different meanings in different contexts, but no confusion should arise. Also, there are differences in notation here from that of Shield (1955*b*).

c	cohesion stress
ϕ	angle of internal friction
ρ_0	initial density
r, θ, z	cylindrical polar co-ordinates
$\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz}$	stress components
u, w	radial and axial velocity components
$\dot{\epsilon}_r, \dot{\epsilon}_\theta, \dot{\epsilon}_z, \dot{\gamma}_{rz}$	strain-rate components
$\sigma_i, \dot{\epsilon}_i\ (i = 1, 2, 3)$	principal stresses and strain-rates
η	angle specifying orientation of principal stresses and strain-rates
$\dot{\lambda}, \mu$	plastic flow-rate parameters
p_a	atmospheric pressure
$c^* = c + p_a \tan\phi$	relative cohesion stress

1. INTRODUCTION

The present investigation in theoretical soil plasticity is concerned with ideal soils whose postulated mechanical behaviour is an approximation to that of a wide class of natural soils. The term *natural soil* as used in this paper is intended to cover the range of the relatively soft uncemented geological materials of the earth's crust, which, broadly speaking, are comparatively loose aggregates of mineral particles, the voids being filled with water or air or both. The distributions of particle size are supposed to be within the clay, silt and sand fractions. The limits of the range of soils envisaged are in a sense represented by frictionless fully saturated clays and cohesionless dry sands, and a partly saturated clay or mixed soil may be regarded as an intermediate case. Specifically excluded from the present discussion are rock-like brittle geological materials of varying degrees of hardness, such as, for example, limestone, for which the present type of plasticity theory is not applicable. The mechanical behaviour of natural soils is both complex and variable and is still imperfectly understood. In this connexion, particularly useful references are the standard texts by Terzaghi (1943) and Terzaghi & Peck (1948) and the review article by Skempton & Bishop (1954). It is not proposed to discuss in detail the physical limitations of the approximation to be adopted here, but some preliminary remarks of a general nature based upon conclusions reached by previous writers on theoretical soil plasticity will now be given.

It seems clear that present studies of theoretical soil plasticity must relate mainly to the phenomenological behaviour of natural soils. In addition, owing to the variable and inhomogeneous physical structure of natural soils, it is necessary to consider ranges of values of ideal soil constants that represent certain averaged rheological properties of natural soils. In certain circumstances (e.g. in some problems of foundation engineering), there appears to be reasonable justification for the adoption of a limit analysis approach based upon Coulomb's (1773) law of failure in soils. This law expresses the mechanical strength of a general soil in terms of certain fairly well defined physical properties, namely, cohesion (due to the bonding action of water and air present between the constituent mineral particles) and friction (due to forces set up at inter-particle contacts). As defined here, the natural soils include clays, silts and sands, a fundamental division existing between those soils that are cohesive substances such as clays and cohesionless particle aggregates such as dry sands. It may be noted that in a fully saturated clay, the inter-granular contacts are effectively lubricated by the pore water, so that the frictional strength is then quite negligible. Of course, in practical applications of theoretical studies based upon a limit analysis approach, considerable care is required in attempting to correlate values, or ranges of values, of the ideal soil constants with the measured values of natural soil constants. Generally in theoretical soil plasticity, the difficulties of procedure are not only of a purely mathematical nature, but are also associated with incompleteness in the present knowledge of the basic physics and also with the paucity of really reliable experimental data. However, in spite of these difficulties, it is hoped that further insight into the basic mechanics of soil plasticity will be achieved through continuing theoretical studies concerned with ideal soils.

The object of the present investigation is to provide a theoretical analysis, valid under certain mathematical and physical assumptions, that has application to a fairly wide class of practical problems of soil mechanics. The class of problems envisaged concerns the general situation of quasi-static axially symmetric plastic flow. Included are problems of load-bearing capacity in foundation engineering and of the mechanical testing of soil samples as in the tri-axial test. Other assumptions made in the present analysis, which relate to the assumed simplified mechanical behaviour of natural soils, are discussed later.

The yield condition on which Coulomb (1773) based his theory of earth pressure includes, as a special case, the yield condition proposed by Tresca (1868) in connexion with the plastic deformation of ductile metals. Interestingly, for reasons of mathematical simplicity, many important contributions currently being made to the mathematical theory of metal plasticity are based directly upon Tresca's yield condition rather than upon von Mises's. Prager (1953) has remarked upon the fact that the early development of the mathematical theory of metal plasticity was strongly influenced, and at times preceded, by developments in the much older theory of earth pressure. Instances of this situation, including some of relatively recent date, are cited by Prager. However, generally, in the more modern developments of the mathematical theory of plasticity, this situation has been reversed. Moreover, inasmuch as the rheological properties of natural soils are more complex and variable than those of metals, it seems quite clear that the development of metal plasticity theory will continue, at least for some time, to lead that of soil plasticity theory. The basic reason for the new situation is that much progress has been made in recent years in the correct formulation of the boundary-value problems of plastic deformation, especially in connexion with metals. Until quite recently, an important defect in the theory of earth pressure lay in its development without reference to stress-strain relations, the theory being based upon the concept of states of limiting equilibrium satisfying Coulomb's law of soil failure in conjunction with a conjectured extremum principle. This procedure altogether neglects the important fact that stress-strain relations are an essential constituent of a complete theory of any branch of the continuum mechanics of deformable solids. Thus, for example, even in a stress boundary-value problem of statical determinacy, although the stress field can be found, at least in principle, without explicit knowledge of an acceptable velocity field, compatibility between these two fields is the essential final justification that the stress field so obtained is in fact the correct one. On the other hand, in a boundary-value problem of statical indeterminacy, there is no such apparent independence between the two fields even in the case of a stress boundary-value problem. These matters are now appreciated in general theory, and are therefore appreciated in relation not only to problems of metal plasticity but also to those of soil plasticity.

It is not intended to give here a complete summary of the origins and development of theoretical soil plasticity. Instead, attention is confined mainly to a brief summary of recent progress from about 1950 onwards. The references given will provide the interested reader with summaries and references to the earlier work. As remarked upon by Hill (1950), the plastic yielding of certain non-metallic materials, e.g. clay and ice, is markedly dependent upon the mean value of the principal stresses. Accordingly, a more elaborate plasticity theory is required for such materials than is required for ductile metals for which there is not this dependence, at least under normal levels of stress intensity. Now at present, investigations based upon the concept of perfect plasticity (i.e. no strain-hardening) form the central and most extensively developed part of the mathematical theory of plasticity. Here, the fundamental importance of von Mises's concept of the plastic potential has been shown in terms of the principle of maximum plastic work, due to Hill (1950), and the theorems of limit analysis, due to Drucker, Greenberg & Prager (see Prager & Hodge 1951). Moreover, it is then possible to prove certain uniqueness theorems. At first, application of the mathematical techniques involved was restricted mainly to the theoretical treatment of problems concerning plastic deformations in ductile metals. However, the corresponding extension to problems involving plastic deformations in soils is more recent, and, in fact, this extension marks the beginning of the development of a consistent mathematical theory of soil plasticity. This proposal was first made by Drucker & Prager (1952), and it has been developed and applied to the discussion of problems of interest in a series of papers by Drucker (1953) and Shield (1953, 1954*a*, *b*, 1955*a*), the general trend being from two-dimensional situations to three-dimensional ones. Prager (1953) has provided some account of the progress made in this field. Finally, it is appropriate to mention here the work of Drucker, Gibson & Henkel (1955) on strain-hardening theories of soil plasticity and also that of Sobotka (1959, 1960) on non-homogeneous soils.

The present paper is concerned with an extension of the application of the techniques of metal plasticity theory to allow the treatment of problems involving plastic deformations in soils. More precisely, a theoretical study is made of quasi-static stress and velocity fields occurring in plastically deformed ideal soils under axially symmetric conditions. The mechanical behaviour attributed to the medium is rigid, perfectly plastic, subject to Coulomb's yield condition and associated flow rule. In suitable circumstances, the behaviour of this ideal soil provides a useful approximation to that of a natural soil, say a typical clay, exhibiting both cohesion and internal friction. It should be particularly noted that the present analysis takes no account of effects due to elastic strain, strain-hardening or inertia. Some account of the effect of soil weight, important under certain circumstances, is included.

The specification of the idealized type of soil mechanical behaviour considered here involves just two parameters, namely, one representing cohesion stress, c ($\geqslant 0$), and the other representing angle of internal friction, ϕ ($0 \leqslant \phi \leqslant \frac{1}{2}\pi$). Consider a small plane element of area δA, of arbitrary orientation, and drawn through a given point within a mass of isotropic cohesive soil. Let $\sigma\,\delta A$ and $\tau\,\delta A$ be the normal and shear forces, respectively, exerted across this element. Then, according to Coulomb's law of plastic flow in soils, the shear stress, τ ($\geqslant 0$), must not exceed an amount that depends linearly upon the cohesion stress and the normal stress, σ (here taken to be positive in tension), i.e.

$$\tau \leqslant c - \sigma \tan \phi. \tag{1·1}$$

On the basis of (1·1), Shield (1955*a*) has correctly formulated the yield condition and associated flow rule appropriate to the general treatment of three-dimensional problems of soil plasticity. In application of the analysis, Shield considered only the approximate solution of the problem of the bearing capacity for a rectangular punch or footing on the plane surface of a semi-infinite mass of soil. It may be noted that this treatment of three-dimensional problems followed upon related work by Drucker (1953), and that J. F. W. Bishop and W. Prager in unpublished work have also independently obtained the essentials of Shield's analysis. In previous work, attention had been confined mainly to the two-dimensional plane strain problems of soil plasticity.

Now Tresca's yield criterion, which applies to ductile metals, corresponds to the particular case of Coulomb's yield criterion when there is no internal friction. In other words, Tresca's law of plastic flow in metals, which is represented by

$$\tau \leqslant k, \tag{1·2}$$

where k (> 0) is the shear yield stress, is alternatively represented by (1·1) with $c = k$ and $\phi = 0$. The condition (1·2) is simply a specialization of (1·1). Fundamentally, Coulomb's yield criterion contrasts with Tresca's in dependence upon the mean value of the principal stresses. Of course, some similarities and, necessarily, general consistency must be expected to occur in theory and applications in analogous situations when Tresca's or Coulomb's yield criterion is adopted. To some extent, and whenever analogous physical situations of interest occur, progress in the solution of soil plasticity problems may be expected to be consequent upon progress in that of metal plasticity problems. In fact, the present paper reflects this situation. Shield (1955*b*) has developed, and applied to some problems of interest, the general theory of axially symmetric plastic flow of rigid, perfectly plastic material obeying Tresca's yield condition and associated flow rule. Detailed analysis revealed considerable variations in the structure of the field equations for the various plastic régimes. On this basis, it was conjectured that those plastic régimes agreeing with the hypothesis of Haar & von Kármán were likely to be of the greatest significance in the solution of problems of interest. This heuristic principle of Haar & von Kármán states, under the present axially symmetric conditions, that the circumferential principal stress is equal to one of the other two principal stresses acting in an axial plane. In this case, the stress and velocity equations are hyperbolic with the same families of characteristics, and, in addition, the stresses are statically determinate. Shield (1955*b*) solved certain problems for which the Haar & von Kármán hypothesis is valid. Recently, Eason & Shield (1960) have extended this earlier work.

Essentially, the present analysis based upon Coulomb's yield criterion for soils may be regarded as a development of Shield's (1955*b*) previous analysis based upon Tresca's yield condition for ductile metals. Although the yield criterion is now dependent upon the mean value of the principal stresses, and the analysis is accordingly more complicated, none of the more basic features of the analysis are thereby affected. An important correction to Shield's analysis should, however, be noted, viz. the velocity fields of two of the plastic régimes are now proved to be hyperbolic, not elliptic (see § 4·2).

Berezancev (1955) has given a discussion of the problems of normal penetration of cohesive soils by a rigid smooth sphere and by a rigid smooth right circular cone. The physical assumptions concerning the mechanical behaviour of the material are the same as those made here. The analysis, however, is open to some adverse criticism. Thus, there is the assumption *a priori* that the stress fields are generated by the Haar & von Kármán plastic régimes. Also, attention is confined to stress fields in plastically deforming regions, and there is no discussion either of the stress fields in rigid regions or of compatible velocity fields. Further, the displacements at the stress-free boundary are entirely neglected. Therefore, the approach made by Berezancev to these soil plasticity problems is, at best, only approximate, and the accuracy of the values obtained for applied loads needed to produce penetration is not known. Shield (1955*b*) has made somewhat similar criticism of work by Ishlinskiĭ (1944) on the problem of the indentation of a semi-infinite body by a circular flat-ended rigid punch, attention being given to material that was rigid, perfectly plastic and that obeyed Tresca's yield criterion. Shield's exact analysis of this problem was required to justify Ishlinskiĭ's method of determination of the applied load needed to produce penetration. However, Shield found Ishlinskiĭ's value of the limit load to be only slightly in error due to the particular numerical procedure adopted. Generally, in situations when penetration occurs, there may very well be quite large displacements at the stress-free boundary, and their neglect may result in serious errors in values of penetration loads.

A most striking feature of the present analysis concerns the almost invariable occurrence of (real) families of characteristics of the stress and velocity equations. This situation stems directly from the adoption of a piece-wise linear yield condition. In Hodge's (1956) terminology, the present paper develops a piece-wise linear isotropic theory of soil plasticity. It is interesting to contrast the present situation with that existing when Tresca's yield criterion, obtaining for $\phi = 0$, is replaced by von Mises's yield criterion. Investigations, by Hill (1950), Parsons (1956) and also other writers, of the equations now governing the stress and velocity fields have shown that there are no real characteristics, except possibly curves along which the radial velocity vanishes. Thus, a marked degree of mathematical simplicity is achieved here through the adoption of Coulomb's yield criterion.

The main content of this paper is set out in three separate parts arranged as follows. Part I concerns the basic equations of quasi-static axially symmetric plastic deformations in the present type of ideal soil. The general field equations are formulated in § 2 and a general discussion of strong discontinuities in the field quantities is given in § 3. Part II concerns the detailed analysis of the basic equations. This analysis is given in § 4, where for convenience the individual plastic régimes are classed as members of four distinct groups. Part III concerns applications of the general theory developed in parts I and II. In § 5, as a preliminary and simple illustrative example, possible velocity fields are obtained for the incipient plastic flow of a right circular cylindrical sample of ideal soil subjected to uni-axial compressive stress parallel to its axis. Next, in § 6, a complete solution is given of the problem of the incipient plastic flow in a semi-infinite region of ideal soil, bounded by a plane surface, due to load applied through a flat-ended, smooth, rigid, circular cylinder. In particular, numerical results are presented for the variation of the yield-point load with angle of internal friction of the ideal soil. Finally, in § 7, concluding remarks on the investigation are made.

PART I. BASIC EQUATIONS

2. General field equations

Let O be the origin of a right-handed system of cylindrical polar co-ordinates r, θ, z (see figure 1). These co-ordinates are Eulerian. A superior dot associated with a quantity denotes convective differentiation, i.e. differentiation following the motion of a particle. Thus, as is usual, $(\ldots)^{\cdot} \equiv (\partial/\partial t + \mathbf{q}\,.\,\nabla)\,(\ldots)$, where $\mathbf{q}$ denotes particle velocity. In the usual notation, let $(\sigma_r, \sigma_\theta, \sigma_z, \tau_{\theta z}, \tau_{rz}, \tau_{r\theta})$ be the stress components, (u, v, w) be the velocity components, and $(\dot{\epsilon}_r, \dot{\epsilon}_\theta, \dot{\epsilon}_z, \dot{\gamma}_{\theta z}, \dot{\gamma}_{rz}, \dot{\gamma}_{r\theta})$ be the strain-rate (tensor) components, referred to this co-ordinate system. The z axis is taken to be the axis of symmetry. Further, if the effect of soil weight is important in any problem considered, then the positive z direction is taken to coincide with the direction in which gravity acts. Axial symmetry requires that the shear stresses $\tau_{\theta z}$ and $\tau_{r\theta}$, the circumferential velocity v and the shear strain-rates $\dot{\gamma}_{\theta z}$ and $\dot{\gamma}_{r\theta}$ all vanish identically, and that all remaining stress, velocity and strain-rate components are functions at most of r, z, t, where t is a suitable time-like co-ordinate. As the mechanical behaviour of the material is assumed to be independent of all reference to physical time, T, it is necessary to consider T explicitly only in dynamical problems when inertial effects are important. In quasi-static problems, to which attention is confined here, a time-scale is merely necessary to order the sequence of events. Accordingly, in the present situation, any suitable monotonic increasing quantity t, correlated with progressive deformation occurring in any particular problem, provides a natural (or built-in) time-scale. Thus, velocity and strain-rate components are conveniently measured with respect to such a time-like co-ordinate t. Of course, the analysis provides only the ratios, and not the absolute values, of such components. This situation simply reflects the fact that no inertial or viscous effects are considered.

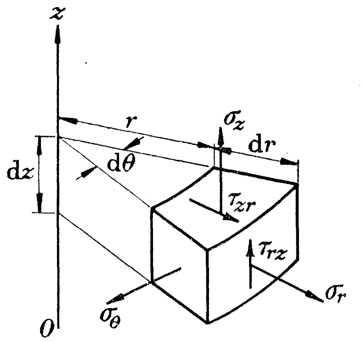

Figure 1. Cylindrical polar co-ordinate system and stress components.

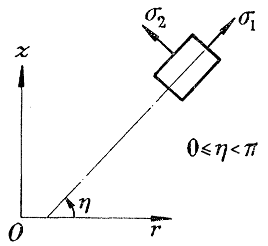

Figure 2. Directions of principal stress.

If quasi-static conditions apply, then the stress components satisfy the equations of equilibrium

$$\frac{\partial \sigma_r}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0, \quad \frac{\partial \tau_{rz}}{\partial r}+\frac{\partial \sigma_z}{\partial z}+\frac{\tau_{rz}}{r}+\rho g=0. \tag{2·1}$$

Now if L and f denote a typical length and acceleration involved in a particular problem, then the condition that inertial effects are sufficiently small to be neglected in the equations as above is that

$$(\rho f L/c, f/g) \ll 1, \tag{2·2}$$

where ρ and c (>0) are the density and cohesion stress of the medium. The components of the strain-rate tensor are

$$\dot{\epsilon}_r=\frac{\partial u}{\partial r}, \quad \dot{\epsilon}_\theta=\frac{u}{r}, \quad \dot{\epsilon}_z=\frac{\partial w}{\partial z}, \quad \dot{\gamma}_{rz}=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right). \tag{2·3}$$

Inasmuch as there is axial symmetry, any axial plane $\theta=$ const. may be taken as a reference plane (r, z), $(r \geqslant 0)$, and all mathematical quantities are to be determined as functions of r, z, t. It should be noted that the condition of axial symmetry imposes certain rather special conditions on the stress, velocity and strain-rate components, and on certain of their derivatives, along the axis. Discussion of these conditions has also been given previously by Hill (1950), Parsons (1956) and other writers. In deriving these conditions, certain assumptions are made. First, plastic flow without fracture is assumed to occur. Secondly, sufficient conditions for the existence of derivatives and other limits are assumed to be satisfied. Now the condition of axial symmetry itself directly requires that

$$\left.\begin{array}{ll}\sigma_r=\sigma_\theta, & \tau_{rz}=0, \\ \dot{\epsilon}_r=\dot{\epsilon}_\theta, & \dot{\gamma}_{rz}=0,\end{array}\right\} \quad \text{on} \quad r=0. \tag{2·4}$$

The condition that particles near the axis in a plane $z=$ const. do not separate is

$$u=0 \quad \text{on} \quad r=0. \tag{2·5}$$

It should be noted, however, that this condition is not always taken to be strictly satisfied in applications to problems. The equilibrium equations (2·1) now show that

$$\lim_{r\to 0}\left(\frac{\partial \sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}\right)=0, \quad \lim_{r\to 0}\left(\frac{\partial \sigma_z}{\partial z}+\frac{2\tau_{rz}}{r}\right)+\rho g=0, \tag{2·6}$$

and the definitions (2·3) show that

$$\lim_{r\to 0}\left(\frac{\partial w}{\partial r}\right)=0. \tag{2·7}$$

Additional relations may of course be obtained through processes of differentiation.

In proceeding, it is necessary now to specify in more detail the mechanical behaviour of the material. All elastic strain is neglected so that effectively the material is completely rigid for stress states below yield. Further, strain-hardening is neglected so there is now no direct restriction on the magnitude of the plastic strain-rates occurring for yield stress states. The plastic strain-rate is taken to be in accord with Coulomb's yield condition and plastic potential, i.e. the von Mises's hypothesis of an associated yield condition and flow rule is adopted. The fundamental relations between stress and strain-rate are most simply and conveniently formulated with reference to the principal stresses and strain-rates,

σ_i and $\dot{\epsilon}_i$ $(i = 1, 2, 3)$, respectively. In the present case of isotropic material, the principal axes of stress and strain-rate coincide. The circumferential direction is automatically a principal direction because of axial symmetry. The conventions $\sigma_3 = \sigma_\theta$ and $\dot{\epsilon}_3 = \dot{\epsilon}_\theta$ are accordingly adopted. Provided that either $\sigma_1 \neq \sigma_2$ or $\dot{\epsilon}_1 \neq \dot{\epsilon}_2$ then the principal directions falling within the reference plane (r, z) are unique. However, these two directions are not yet distinguished between, nor is their orientation specified to within an integral multiple of π. It is required to introduce a definite system of right-handed co-ordinate axes associated with these principal directions. The conventions $\sigma_1 \geqslant \sigma_2$ and (equivalently, because of isotropy) $\dot{\epsilon}_1 \geqslant \dot{\epsilon}_2$ are adopted, and then the positive, first and second, principal directions are defined to make an angle of η $(0 \leqslant \eta < \pi)$ with the positive r and z directions, respectively (see figure 2). The basic equations of transformation of stress components are then

$$\left.\begin{aligned} \sigma_1 &= \tfrac{1}{2}(\sigma_r+\sigma_z)+\{\tfrac{1}{4}(\sigma_r-\sigma_z)^2+\tau_{rz}^2\}^{\frac{1}{2}}, \\ \sigma_2 &= \tfrac{1}{2}(\sigma_r+\sigma_z)-\{\tfrac{1}{4}(\sigma_r-\sigma_z)^2+\tau_{rz}^2\}^{\frac{1}{2}}, \\ \sigma_3 &= \sigma_\theta, \end{aligned}\right\} \tag{2·8}$$

and, conversely,

$$\left.\begin{aligned} \sigma_r &= \tfrac{1}{2}(\sigma_1+\sigma_2)+\tfrac{1}{2}(\sigma_1-\sigma_2)\cos 2\eta, \\ \sigma_z &= \tfrac{1}{2}(\sigma_1+\sigma_2)-\tfrac{1}{2}(\sigma_1-\sigma_2)\cos 2\eta, \\ \tau_{rz} &= \tfrac{1}{2}(\sigma_1-\sigma_2)\sin 2\eta, \end{aligned}\right\} \tag{2·9}$$

where $$\cos 2\eta/\tfrac{1}{2}(\sigma_r-\sigma_z) = \sin 2\eta/\tau_{rz} = 1/\{\tfrac{1}{4}(\sigma_r-\sigma_z)^2+\tau_{rz}^2\}^{\frac{1}{2}}, \quad (0 \leqslant \eta < \pi). \tag{2·10}$$

The analogous equations involving the strain-rate components are

$$\left.\begin{aligned} \dot{\epsilon}_1 &= \tfrac{1}{2}(\dot{\epsilon}_r+\dot{\epsilon}_z)+\{\tfrac{1}{4}(\dot{\epsilon}_r-\dot{\epsilon}_z)^2+\dot{\gamma}_{rz}^2\}^{\frac{1}{2}}, \\ \dot{\epsilon}_2 &= \tfrac{1}{2}(\dot{\epsilon}_r+\dot{\epsilon}_z)-\{\tfrac{1}{4}(\dot{\epsilon}_r-\dot{\epsilon}_z)^2+\dot{\gamma}_{rz}^2\}^{\frac{1}{2}}, \\ \dot{\epsilon}_3 &= \dot{\epsilon}_\theta, \end{aligned}\right\} \tag{2·11}$$

and, conversely,

$$\left.\begin{aligned} \dot{\epsilon}_r &= \tfrac{1}{2}(\dot{\epsilon}_1+\dot{\epsilon}_2)+\tfrac{1}{2}(\dot{\epsilon}_1-\dot{\epsilon}_2)\cos 2\eta, \\ \dot{\epsilon}_z &= \tfrac{1}{2}(\dot{\epsilon}_1+\dot{\epsilon}_2)-\tfrac{1}{2}(\dot{\epsilon}_1-\dot{\epsilon}_2)\cos 2\eta, \\ \dot{\gamma}_{rz} &= \tfrac{1}{2}(\dot{\epsilon}_1-\dot{\epsilon}_2)\sin 2\eta, \end{aligned}\right\} \tag{2·12}$$

where $$\cos 2\eta/\tfrac{1}{2}(\dot{\epsilon}_r-\dot{\epsilon}_z) = \sin 2\eta/\dot{\gamma}_{rz} = 1/\{\tfrac{1}{4}(\dot{\epsilon}_r-\dot{\epsilon}_z)^2+\dot{\gamma}_{rz}^2\}^{\frac{1}{2}}, \quad (0 \leqslant \eta < \pi). \tag{2·13}$$

It follows from (2·10) and (2·13) that

$$(\sigma_r-\sigma_z)/(\dot{\epsilon}_r-\dot{\epsilon}_z) = \tau_{rz}/\dot{\gamma}_{rz}. \tag{2·14}$$

The relation (2·14) is a necessary, but not sufficient, condition for isotropy. However, from (2·9) and (2·13), it follows that both ratios (2·14) are non-negative and equal to $(\sigma_1-\sigma_2)/(\dot{\epsilon}_1-\dot{\epsilon}_2)$, and this is the necessary and sufficient condition for isotropy.

Now Shield (1955*a*) has given the correct formulation of Coulomb's yield condition and associated flow rule under quite general conditions of three-dimensional stress. If the (algebraic) maximum, intermediate and minimum principal stresses are σ_{I}, σ_{II} and σ_{III}, then the yield condition is

$$\sigma_{\mathrm{I}}-\sigma_{\mathrm{III}} = 2c\cos\phi-(\sigma_{\mathrm{I}}+\sigma_{\mathrm{III}})\sin\phi, \quad (\sigma_{\mathrm{I}} \geqslant \sigma_{\mathrm{II}} \geqslant \sigma_{\mathrm{III}}), \tag{2·15}$$

where c and ϕ are physical parameters of the material, denoting cohesion stress and angle of internal friction, respectively. Note that under isotropic stress conditions,

$\sigma_{\mathrm{I}} = \sigma_{\mathrm{II}} = \sigma_{\mathrm{III}} = c\cot\phi$. The case $\phi = 0$ and $c > 0$ corresponds to a frictionless material, representative say of a fully saturated clay, and the case $c = 0$ and $\phi > 0$ corresponds to a cohesionless material, representative say of a dry sand. The intermediate case of a material exhibiting both cohesion and internal friction is representative of a clay for which $c > 0$ and (typically) $0° < \phi < 40°$. For definiteness, the restrictions

$$c > 0, \quad 0 \leqslant \phi < \tfrac{1}{2}\pi \tag{2·16}$$

are generally adopted here. The case $c = 0$ and $\phi > 0$ needs special consideration which is not given in detail here.

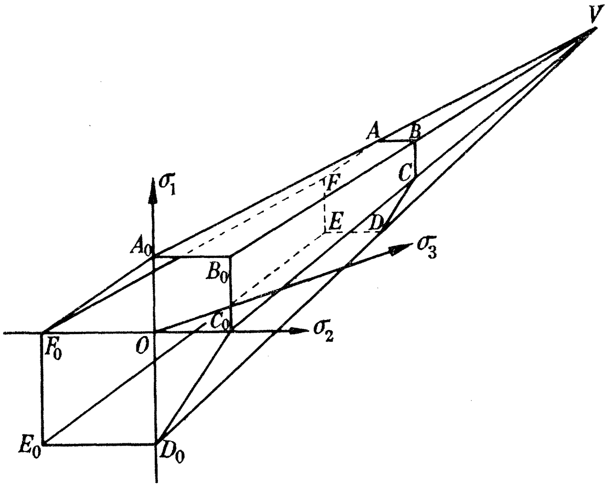

FIGURE 3. Yield surface in principal stress space.

The yield condition (2·15) is most simply represented by a surface drawn in a hypothetical three-dimensional space in which a stress state σ_i $(i = 1, 2, 3)$ is represented by a point whose co-ordinates with respect to a rectangular Cartesian frame of reference are σ_i $(i = 1, 2, 3)$. In figure 3, the yield surface is shown as an irregular hexagonal (double) right pyramid with vertex V at the point $\sigma_i = c\cot\phi > 0$ $(i = 1, 2, 3)$ and with axis as the line through the co-ordinate origin O and V. The cross-section of this pyramid by a general plane $\sigma_3 =$ const. is the irregular hexagon $ABCDEF$. The six edges of the pyramid pass through V and the points $A_0(a_0, 0, 0)$, $B_0(a_0, a_0, 0)$, $C_0(0, a_0, 0)$, $D_0(-b_0, 0, 0)$, $E_0(-b_0, -b_0, 0)$ and $F_0(0, -b_0, 0)$, where

$$\left.\begin{aligned} a_0/2c &= \tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\phi\right) = \cos\phi/(1+\sin\phi), \\ b_0/2c &= \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right) = \cos\phi/(1-\sin\phi), \\ \text{and hence} \qquad (a_0/2c)\,(b_0/2c) &= 1, \quad 1/a_0 - 1/b_0 = (1/c)\tan\phi. \end{aligned}\right\} \tag{2·17}$$

In the present paper, no explicit attention will be given to that part of the yield surface either at or beyond the vertex V, i.e. the restriction

$$\sigma_i < c\cot\phi \quad (i = 1, 2, 3) \tag{2·18}$$

is adopted. This restriction seems likely to be well satisfied in certain classes of problems, e.g. penetration problems, which are characterized by stress conditions that are generally of a compressive, rather than a tensile, nature. The restriction may in fact involve little limitation in practical applications of the analysis, but in any event, it is easily relaxed should the need arise. Further, it is quite straightforward to give attention to other restrictions on the yield surface corresponding to *tension cut-offs* (see Shield 1955*a*). The complete yield surface is shown in figure 3, stress states below yield corresponding to points inside the pyramid *VABCDEF* and those at yield corresponding to points on this pyramid. In other words, points on the varying hexagon *ABCDEF* represent all possible plastic stress states given by (2·15). Now according to von Mises's theory of the plastic potential, the principal plastic strain-rate vector $\dot{\boldsymbol{\epsilon}}(\dot{\epsilon}_i)$, associated with a principal stress vector $\boldsymbol{\sigma}(\sigma_i)$, is directed along the normal drawn outwards to the yield surface at the point σ_i $(i = 1, 2, 3)$. This direction is unique except at the singular points along the edges $V(A, B, C, D, E, F)$. In such a case, following the Koiter–Prager generalization of von Mises's rule, the direction $\dot{\boldsymbol{\epsilon}}$ is restricted merely to lie between, and in the plane defined by, the unique normals drawn outwards to the two faces of the pyramid intersecting in the particular edge considered. Thus, if $f(\sigma_1, \sigma_2, \sigma_3) = 0$ generally denotes the yield condition, with $f < 0$ corresponding to stress states below yield, then formally

$$\dot{\epsilon}_i = \left\{\begin{array}{ll} \dot{\lambda}\dfrac{\partial f}{\partial \sigma_i} & \text{if} \quad f = \dot{f} = 0, \\ 0 & \text{if} \quad \text{either (i) } f < 0 \quad \text{or} \quad \text{(ii) } f = 0 \text{ and } \dot{f} < 0, \end{array}\right\} \quad \dot{\lambda} \geqslant 0, \quad (i = 1, 2, 3), \quad (2{\cdot}19)$$

$\dot{\lambda}$ being a scalar non-negative plastic flow-rate parameter of position and time, and the differential coefficients $\partial f/\partial \sigma_i$ being correctly interpreted at singular points. In view of the previous (notational) restriction that $\sigma_1 \geqslant \sigma_2$, it follows that only one-half, namely *VEFAB*, of the complete yield surface is to be considered, there being symmetry of course about the plane *VBE*. For simplicity, reference will be made to the various plastic régimes involved in four distinct natural groups, individual plastic régimes being denoted simply by reference to vertices or sides of *EFAB*, namely: I, *B* and *E*; II, *AB* and *EF*; III, *A* and *F*; and IV, *AF*. Thus, the groups I and III comprise the singular edge plastic régimes, whereas the groups II and IV comprise the regular face plastic régimes. The yield functions f_{PQ} corresponding to the face plastic régimes $PQ = (DE, EF, FA, AB, BC)$ are

$$\left.\begin{aligned} f_{DE} &= -\sigma_1/b_0 + \sigma_3/a_0 - 1, \\ f_{EF} &= -\sigma_2/b_0 + \sigma_3/a_0 - 1, \\ f_{FA} &= \sigma_1/a_0 - \sigma_2/b_0 - 1, \\ f_{AB} &= \sigma_1/a_0 - \sigma_3/b_0 - 1, \\ f_{BC} &= \sigma_2/a_0 - \sigma_3/b_0 - 1. \end{aligned}\right\} \qquad (2{\cdot}20)$$

The flow rules (2·21) (table 1) for the various plastic régimes are now simply obtained from (2·19) and (2·20). In the case of the singular plastic régimes, it is convenient to introduce a scalar parameter μ $(0 \leqslant \mu \leqslant 1$ at most) of position and time whose increase corresponds to a transition between the relevant regular plastic régimes reckoned in an anti-clockwise sense round the hexagon *ABCDEF*. Here, the edge and face plastic régimes are defined as closed régimes, i.e. to have coincident termini.

Table 1. Yield conditions and flow rules for individual plastic régimes (2·21)

group	plastic régime	yield condition. stress state. μ	$\dot{\epsilon}_1/\dot{\lambda}$	$\dot{\epsilon}_2/\dot{\lambda}$	$\dot{\epsilon}_3/\dot{\lambda}$
I	B	$\sigma_1=\sigma_2=a_0+\sigma_3/N$; $\sigma_1=\sigma_2>\sigma_3$; $0\leqslant\mu\leqslant\frac{1}{2}$	$(1-\mu)/a_0$	μ/a_0	$-1/b_0$
	E	$\sigma_1=\sigma_2=-b_0+N\sigma_3$; $\sigma_3>\sigma_1=\sigma_2$; $\frac{1}{2}\leqslant\mu\leqslant 1$	$-(1-\mu)/b_0$	$-\mu/b_0$	$1/a_0$
II	AB	$\sigma_1=a_0+\sigma_3/N$; $\sigma_1\geqslant\sigma_2\geqslant\sigma_3$	$1/a_0$	0	$-1/b_0$
	EF	$\sigma_2=-b_0+N\sigma_3$; $\sigma_3\geqslant\sigma_1\geqslant\sigma_2$	0	$-1/b_0$	$1/a_0$
III	A	$\sigma_2=\sigma_3=N(\sigma_1-a_0)$; $\sigma_1>\sigma_2=\sigma_3$; $0\leqslant\mu\leqslant 1$	$1/a_0$	$-(1-\mu)/b_0$	$-\mu/b_0$
	F	$\sigma_1=\sigma_3=(\sigma_2+b_0)/N$; $\sigma_1=\sigma_3>\sigma_2$; $0\leqslant\mu\leqslant 1$	μ/a_0	$-1/b_0$	$(1-\mu)/a_0$
IV	FA	$\sigma_1/a_0-\sigma_2/b_0=1$; $\sigma_1\geqslant\sigma_3\geqslant\sigma_2$	$1/a_0$	$-1/b_0$	0

$$N=N(\phi)=\tan^2(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi)\geqslant 1$$

It should be noted that the result

$$\dot{\epsilon}_1+\dot{\epsilon}_2+\dot{\epsilon}_3=(1/a_0-1/b_0)\,\dot{\lambda}=(\dot{\lambda}/c)\tan\phi\geqslant 0 \tag{2·22}$$

holds quite generally at points on the yield surface, and hence plastic flow is accompanied by dilatation, and therefore by decrease in density of the medium, except in the special case $\phi=0$. Now the equation of conservation of mass is

$$\dot{\rho}+\rho(\dot{\epsilon}_1+\dot{\epsilon}_2+\dot{\epsilon}_3)=0, \tag{2·23}$$

and hence from (2·22),

$$\dot{\rho}=-\rho(\dot{\lambda}/c)\tan\phi\leqslant 0. \tag{2·24}$$

The relation (2·24) may be formally integrated to give

$$\rho=\rho_0\exp\{-(\lambda/c)\tan\phi\} \tag{2·25}$$

for a particular particle. This result shows quite clearly that any amount of plastic strain λ (>0) always corresponds to a decrease in density unless $\phi=0$. In any problem of continuing plastic flow in which the weight of the soil is important, the analysis is complicated by density changes that occur as the deformation proceeds. However, in any problem of incipient plastic flow, $\rho=\rho_0$ in (2·1) and $\dot{\rho}$ is simply given by (2·24). It should also be noted that at large plastic strain, (2·25) implies considerable reductions in ρ, assuming that $\phi>0$. Now this situation is unlikely from a physical standpoint for, although natural soils do exhibit some degree of dilatancy, this is strictly limited, and relatively small density changes due to internal friction only occur. It must therefore be concluded that the assumption of Coulomb's yield condition and flow rule is not a valid one in the analysis of a problem involving large plastic strain. In the present paper, application is envisaged either to problems of incipient plastic flow in which effects due to soil weight are important, or to problems of continuing plastic flow in which such effects are neglected. The condition for effects due to soil weight to be sufficiently small to be neglected is that

$$\rho gL/c\ll 1, \tag{2·26}$$

this being a restriction additional to (2·2).

The general field equations are therefore given by (2·1), (2·21) and (2·24).

3. Strong discontinuities in field quantities

In this section, a discussion will be given of space-discontinuities (or jumps) in the velocity and stress components and the density. A comprehensive treatment would be lengthy and will not be attempted here. Moreover, it seems preferable to confine attention initially to fundamentals and to leave detailed aspects for later consideration directly in relation to the actual solution of problems.

The analysis of the field equations, developed generally in § 2 (specialized later in § 4 to the various individual plastic régimes) implicitly assumes certain conditions of continuity and differentiability on the velocity and stress components and the density. In other words, the concern there is to study stress and velocity fields on the supposition that discontinuities only of a weak nature are perhaps involved. However, in any particular problem, strong discontinuities in the stress and velocity fields may be involved, and in these circumstances at least some of the fundamental differential equations must be replaced locally by finite relations. As will be seen, such discontinuities can arise in a variety of ways. In general, of course, special conditions must be satisfied if postulated discontinuities are actually to occur. The main purpose of the present discussion is to show how the permissible discontinuities and associated conditions are determined.

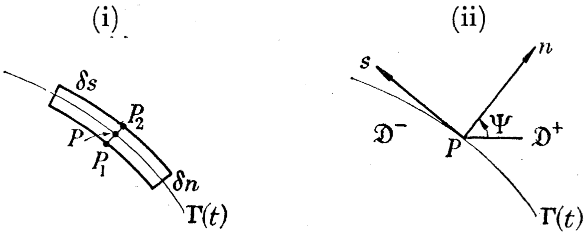

Figure 4. Space-discontinuities.

Under the present restriction to axially symmetric conditions, any possible jumps in physical quantities must necessarily take place across curves in an axial plane. Let $\Gamma(t)$ be a simple open curve (see figure 4 (i)), in general varying with the time, drawn in an axial plane. Suppose that an isolated finite jump in at least one field quantity, say G, occurs everywhere along Γ. Such a discontinuity is to be regarded mathematically as the limit of a continuous distribution that changes by a certain amount across a narrow region enclosing Γ as this region everywhere shrinks up to Γ. It is convenient to think of this region as a strip with its edges either parallel or normal to Γ. The amount of the jump in G across Γ is defined by

$$\left.\begin{aligned}[G] &= G^+ - G^-,\\ \text{where}\qquad G^\pm &= \lim G(P_{2,1}) \quad \text{as} \quad P_{2,1} \to P,\end{aligned}\right\} \tag{3·1}$$

in the notation of figure 4 (i).

Prager (1954) has given a systematic discussion and classification of discontinuities in (generalized) stress and velocity fields occurring in non-hardening plastic, rigid continua. Of course, certain discontinuities in physical quantities cannot be precisely true in a natural solid material. Strictly, various aspects of mechanical behaviour, such as effects due to elastic strain, strain-hardening and viscosity, all of which are neglected here, will act to diffuse sharp discontinuities into a gradual transition over some region. The physical interpretation of discontinuities occurring in ideal continua obviously requires considerable care.

For simplicity, the arguments of the present section will be largely formal, the exact statement of analytical conditions sufficient for the validity of the results being omitted.

It should be noted first that there are kinematical restrictions on permissible jumps in the derivatives of a continuous quantity. Now let n, s be local rectangular Cartesian coordinates, respectively normal and tangential to Γ at a point P, as shown in figure 4 (ii). Here, it is found convenient to regard G as a function of the variables n, s, t. The equation to Γ is taken to be $g(n,s,t) = 0$, where the value of g increases in the normal direction. Suppose that G is continuous across Γ, i.e. $[G] = 0$. Then it is straightforward to prove that

$$\left.\begin{aligned} &[\partial G/\partial s] = 0, \quad V[\partial G/\partial n] + [\partial G/\partial t] = 0, \\ \text{where} \qquad &V = -\frac{\partial g}{\partial t}\bigg/\frac{\partial g}{\partial n} \end{aligned}\right\} \tag{3·2}$$

is the normal speed of propagation of $\Gamma(t)$ at $P(t)$ (see, for example, Hopkins 1957). Thus, $\partial G/\partial s$ is always proved continuous. In respect of $\partial G/\partial n$ and $\partial G/\partial t$, however, two cases arise, namely, (i) if $V = 0$, then $\partial G/\partial t$ is proved continuous but $\partial G/\partial n$ may be discontinuous, and (ii) if $V \neq 0$, then $\partial G/\partial n$ and $\partial G/\partial t$ may be either continuous or discontinuous together. Of course, whenever the argument establishes the continuity of a derivative of G, a second application establishes further results.

Now space-discontinuities in particle displacement must correspond to fracture and must therefore normally be excluded. However, as will be seen, space-discontinuities in particle velocity, stress and density may occur. In proceeding, it is necessary to make use of certain physical laws. The general dynamic case will be considered first, and the quasi-static case will be properly regarded as a specialization. The arguments involved closely parallel those employed in fluid mechanics (see, for example, Howarth 1953). In order to obtain the discontinuity conditions, it is simplest to consider the motion relative to Γ in the immediate vicinity of the point P being considered. Suffixes n, s are used to denote components of quantities with respect to the frame $P(n,s)$. Let $\mathbf{n}$, $\mathbf{s}$ be unit vectors along the positive n, s directions, respectively. Then the particle velocity is $\mathbf{q} = v_s\mathbf{s} + v_n\mathbf{n}$. Consider an elementary region $\delta s \times \delta n$ where $\delta s \gg \delta n$ as shown in figure 4 (i). The application of the basic laws of mechanics to this element of material, allowing δn to tend to zero and δs to become arbitrarily small, provides the required jump conditions on physical quantities. First, the law of conservation of mass requires the rates of inflow and outflow of mass to balance. This condition leads to

$$[m] = 0, \quad \text{where} \quad m = \rho(v_n - V), \tag{3·3}$$

so that the mass flux m is unchanged across Γ. Secondly, the rate of change in linear momentum of the particles passing through the element is equal to the external forces acting on the element. This condition leads finally to

$$m[\mathbf{q}] = [\mathbf{\Sigma}], \quad \text{where} \quad \mathbf{\Sigma} = \sigma_n\mathbf{n} + \tau_{ns}\mathbf{s}, \tag{3·4}$$

in view of (3·3). The application to the element of the laws of conservation of energy and of non-decrease in entropy would provide further results, but these results are not directly of interest here as attention is merely being given to a limited description of the thermodynamical state. Suppose now that quasi-static conditions apply so that the motion proceeds indefinitely slowly. More precisely, if v and σ are a typical particle velocity and stress, it is assumed under quasi-static conditions that

$$\rho_0 v^2/\sigma \ll 1. \tag{3·5}$$

The physical conditions (3·3, 4) on jumps now simplify to

$$[\rho(v_n - V)] = 0 \tag{3·6}$$

and

$$[\sigma_n] = 0, \quad [\tau_{ns}] = 0. \tag{3·7}$$

Now no restrictions have been established here on possible jumps either in the tangential velocity component v_s or in the direct stress components σ_s and σ_θ, and these quantities may possibly be (although are not necessarily) discontinuous.

The fundamental results necessary for the discussion of jumps in physical quantities are contained in (3·2, 6, 7). The cases that arise may be classified in the following manner. Let $\mathfrak{D}^{\pm}$ denote the two regions $n \gtrless 0$ separated by Γ (see figure 4 (ii)). First, either (i) $\mathfrak{D}^{\pm}$ are both below yield, or (ii) $\mathfrak{D}^{-}$ (say) is below yield and $\mathfrak{D}^{+}$ is at yield, or (iii) $\mathfrak{D}^{\pm}$ are both at yield. Of course, if a region is below yield, then it is rigid; but, on the other hand, if a region is at yield, then it is not necessarily deforming plastically. Clearly, the cases (ii) and (iii) which involve plastic stress and velocity fields are of principal interest. Secondly, these cases may be subdivided according to the plastic régimes involved. In case (iii), different plastic régimes may apply in $\mathfrak{D}^{\pm}$. Thus, there are a considerable number of different circumstances that would require attention in any complete discussion. If there are points of intersection between two or more boundaries Γ, then special attention must be given to conditions at such points (cf. Winzer & Carrier 1948).

Attention here, however, will be confined merely to a discussion of one particular matter of obvious interest. Consider the implications of velocity discontinuities. In the special case when $\phi = 0$, (2·24) shows that ρ is constant, so that, from (3·3), $[v_n] = 0$, and hence the only possible discontinuity is $[v_s] \neq 0$. In this event, $\dot{\gamma}_{ns} = \frac{1}{2}(\partial v_n/\partial s + \partial v_s/\partial n)$ is indefinitely larger than both $\dot{\epsilon}_n = \partial v_n/\partial n$ and $\dot{\epsilon}_s = \partial v_s/\partial s$, and therefore Γ must bisect the directions of principal strain-rate. It follows from the flow rule relations given in (2·21) that such a discontinuity is possible only for plastic régimes A, F and FA, i.e. groups III and IV. In the general case when $\phi \neq 0$, there are three possibilities, viz. (i) $[v_n] \neq 0$ and $[v_s] = 0$, (ii) $[v_n] = 0$ and $[v_s] \neq 0$, and (iii) $[v_n, v_s] \neq 0$. With use of the flow rule relations (2·21), (i) and (ii) can be shown not to be possible with a finite value of $\dot{\epsilon}_3 = u/r$. Thus, the only possible type of velocity discontinuity in a frictional medium is one in which both v_n and v_s are discontinuous, implying that incipient separation of adjacent lines of particles occurs. Further, it appears from (2·21) that such a discontinuity is possible only for plastic régimes A, F and FA, i.e. groups III and IV. In addition, it may be shown that $[v_n] = [v_s] \tan\phi$ and that Γ must make an angle $\pm(\frac{1}{4}\pi + \frac{1}{2}\phi)$ with the first principal strain-rate direction.

The discussion of stress discontinuities is best considered following the more detailed study of the governing equations (such discussion is given later at the end of § 4·3 for plastic régimes A and F).

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

PART II. DETAILED THEORETICAL ANALYSIS

4. Analysis of field equations for individual plastic régimes

The general field equations derived in § 2 will now be specialized to the various plastic régimes of groups I to IV and an analysis will be given of their mathematical structure.

Anticipating the conclusions to be drawn in § 4·5, wherein is given a summary of the results obtained in §§ 4·1–4, it may be stated that the plastic régimes of group III are conjectured to be of major importance in practical applications of the theory. For this reason, the discussion of these régimes given in § 4·3 is more detailed than that given for the other régimes.

Groups I to III each involve two plastic régimes, but nevertheless a single analysis suffices for both cases.

4·1. *Group I. Plastic régimes B and E*

The plastic régimes B and E comprising group I are *semi-isotropic* and singular, the former corresponding to a higher mean value of the principal stresses than does the latter, for a given value of σ_θ. In both plastic régimes, the condition

$$\sigma_1 = \sigma_2 \tag{4·1·1}$$

holds and therefore (see (2·9))

$$\sigma_1 = \sigma_2 = \sigma_r = \sigma_z, \quad \tau_{rz} = 0. \tag{4·1·2}$$

In proceeding, it is convenient to characterize the two plastic régimes B and E respectively by the introduction of a pure number ϖ equal to $+1$ or -1. Then, in either case, the complete yield condition given by (2·21) is now

$$\left.\begin{aligned} \sigma_r = \sigma_z &= (Y+\sigma_\theta)/k, \\ \text{where} \qquad k = \frac{1+\varpi\sin\phi}{1-\varpi\sin\phi}&, \quad Y = \frac{2\varpi c\cos\phi}{1-\varpi\sin\phi}. \end{aligned}\right\} \tag{4·1·3}$$

Substitution of (4·1, 2, 3) into the equations of equilibrium (2·1) shows that σ_r satisfies the equations

$$\left.\begin{aligned} \frac{\partial\sigma_r}{\partial r}+(1-k)\frac{\sigma_r}{r}+\frac{Y}{r} &= 0, \\ \frac{\partial}{\partial z}(\sigma_r+\rho g z) &= 0. \end{aligned}\right\} \tag{4·1·4}$$

If $\phi > 0$ (i.e. $k \neq 1$), then (4·1·4) admit no solution unless the soil-weight term $\rho g z$ is neglected. On the other hand, if $\phi = 0$, then (4·1·4) admit a general solution. In the first case, it can be shown that

$$\left.\begin{aligned} \sigma_r = \sigma_z &= c\cot\phi + D_1 r^{k-1}, \\ \sigma_\theta &= c\cot\phi + D_1 k r^{k-1}, \end{aligned}\right\} \quad \phi > 0, \ \rho g L/c \ll 1, \tag{4·1·5}$$

where D_1 $(< 0$ because of (2·18)) is an arbitrary constant, whereas in the second case

$$\left.\begin{aligned} \sigma_r = \sigma_z &= -2\varpi c\ln r - \rho g z + D_2, \\ \sigma_\theta &= -2\varpi c(1+\ln r) - \rho g z + D_2, \end{aligned}\right\} \quad \phi = 0, \tag{4·1·6}$$

where D_2 is an unrestricted arbitrary constant. The occurrence of certain singularities in the stress field at $r = 0$ may be noted.

Elimination of the parameters λ and μ from the flow rule (2·21) provides one relation between the principal strain-rates. Since the directions of principal stress are not unique, the usual condition of isotropy fails to provide a further explicit relation between the principal strain-rates. In general, therefore, the velocity field is not determinate, being governed only by the single equation

$$\left.\begin{aligned} &\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}+\frac{ku}{r}=0,\\ \text{and the conditions}\quad &\varpi(\dot{\epsilon}_1,\dot{\epsilon}_2,-\dot{\epsilon}_3)\geqslant 0,\\ \text{where}\quad &(\dot{\epsilon}_1,\dot{\epsilon}_2,\dot{\epsilon}_3)=\left\{(\max,\min)\left(\frac{\partial u}{\partial r},\frac{\partial w}{\partial z}\right),\frac{u}{r}\right\}.\end{aligned}\right\} \tag{4·1·7}$$

The radial velocity component u is therefore uniformly negative or positive for the plastic régimes B and E, respectively.

4·2. *Group II. Plastic régimes AB and EF*

The two plastic régimes AB and EF comprising group II are regular, the former corresponding to a higher mean value of the principal stresses than does the latter, for a given value of σ_θ. These plastic régimes are characterized essentially by the vanishing of either one of the two principal strain-rates in an axial plane, i.e. one of $\dot{\epsilon}_1$ and $\dot{\epsilon}_2$ is zero (see (2·21)). A second relation between the strain-rates follows from the elimination of the parameter λ occurring in the flow rule (2·21).

In proceeding, it is convenient to characterize the two plastic régimes AB and EF, respectively, through the introduction of a pure number ϖ with values $+1$ and -1. Then, in both cases, equations (2·11, 21) show that the velocity field is governed by

$$\left.\begin{aligned} -k\frac{u}{r}=\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}&=\varpi\left\{\left(\frac{\partial u}{\partial r}-\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\right\}^{\frac{1}{2}},\\ \text{where}\quad k&=\frac{1+\varpi\sin\phi}{1-\varpi\sin\phi}.\end{aligned}\right\} \tag{4·2·1}$$

As $\varpi = \operatorname{sgn}(-u)$, it follows that the radial velocity component is uniformly negative or positive according as plastic régime AB or EF applies. Thus, descriptively, *waisting* or *barrelling* may be said to occur in axial planes.

It is apparent from equations (4·2·1) that the velocity field is kinematically determinate (see § 4·5) and the structure of these equations will now be examined.

Now the first equation (4·2·1) may be written in the form

$$\frac{\partial}{\partial r}(r^k u)+\frac{\partial}{\partial z}(r^k w)=0,$$

so that the velocity components u, w are expressible in terms of a velocity function $V(r,z)$, namely

$$u=r^{-k}\frac{\partial V}{\partial z},\quad w=-r^{-k}\frac{\partial V}{\partial r}. \tag{4·2·2}$$

The equation satisfied by V is, from the second equation (4·2·1),

$$\left(\frac{\partial^2 V}{\partial r^2}-\frac{\partial^2 V}{\partial z^2}-\frac{k}{r}\frac{\partial V}{\partial r}\right)^2+4\frac{\partial^2 V}{\partial r\partial z}\left(\frac{\partial^2 V}{\partial r\partial z}-\frac{k}{r}\frac{\partial V}{\partial z}\right)=0. \tag{4·2·3}$$

Equation (4·2·3) is a non-linear second-order partial differential equation for V. In order to find the nature of this equation, suppose that u and w (i.e. $\partial V/\partial r$ and $\partial V/\partial z$) are given on some curve Γ and it is required to find u and w in the neighbourhood of this curve. From equation (4·2·3), together with the equations expressing the known variation of $\partial V/\partial r$ and $\partial V/\partial z$ along Γ, namely

$$\left.\begin{aligned} \mathrm{d}\left(\frac{\partial V}{\partial r}\right) &= \frac{\partial^2 V}{\partial r^2}\,\mathrm{d}r+\frac{\partial^2 V}{\partial r\,\partial z}\,\mathrm{d}z,\\ \mathrm{d}\left(\frac{\partial V}{\partial z}\right) &= \frac{\partial^2 V}{\partial r\,\partial z}\,\mathrm{d}r+\frac{\partial^2 V}{\partial z^2}\,\mathrm{d}z, \end{aligned}\right\} \tag{4·2·4}$$

the second-order differential coefficients $\partial^2V/\partial r^2$, $\partial^2 V/\partial r\,\partial z$, $\partial^2 V/\partial z^2$ can always be found, provided that the problem is well set. However, from the equations corresponding to (4·2·4) for the variation of these last differential coefficients along Γ, together with the differential form of (4·2·3), the third-order differential coefficients of V cannot be determined uniquely if the curve Γ is such that along Γ,

$$\left.\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}z}{\mathrm{d}r}\left(\frac{\partial^2 V}{\partial z^2}-\frac{\partial^2 V}{\partial r^2}+\frac{k}{r}\frac{\partial V}{\partial r}\right) = -\frac{\partial^2 V}{\partial r\,\partial z}+\frac{k}{r}\frac{\partial V}{\partial z}, \quad -\frac{\partial^2 V}{\partial r\,\partial z},\\ &\text{or, using (2·12, 2·21, 4·2·2),} \qquad \mathrm{d}z/\mathrm{d}r = \tan\eta, \quad -\cot\eta. \end{aligned}\right\} \tag{4·2·5}$$

In other words, (4·2·3) is hyperbolic with characteristic directions given by (4·2·5), namely the directions of principal strain-rate.

Attention will now be given to the structure of the equations governing the stress field. The two equilibrium equations (2·1), the yield condition (2·21) and, since η is known from the solution of the velocity field, the isotropy condition (2·14) are available to determine σ_r, σ_θ, σ_z and τ_{rz}, namely

$$\left.\begin{aligned} \frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r} &= 0,\\ \frac{\partial\tau_{rz}}{\partial r}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}+\rho g &= 0,\\ (\sigma_\theta+Y)/k &= \tfrac{1}{2}(\sigma_r+\sigma_z)+\varpi\{\tfrac{1}{4}(\sigma_r-\sigma_z)^2+\tau_{rz}^2\}^{\frac{1}{2}},\\ \sigma_r-\sigma_z &= 2\tau_{rz}\cot 2\eta,\\ \text{where}\qquad Y &= \frac{2\varpi c\cos\phi}{1-\varpi\sin\phi}. \end{aligned}\right\} \tag{4·2·6}$$

From the second equation (4·2·6) it is apparent that τ_{rz} and σ_z, and hence, with use of the third and fourth equations, that σ_r and σ_θ can all be expressed in terms of a stress function $\Phi(r, z)$. Substitution of these expressions into the first equation leads to a second-order linear partial differential equation for Φ which is easily shown to be hyperbolic and to have the same characteristics as the equation (4·2·3) for the velocity function V.

4·3. *Group III. Plastic régimes A and F*

The two plastic régimes A and F comprising group III are singular, the former corresponding to a higher mean value of the principal stresses than does the latter, for a given value of σ_θ. These plastic régimes are characterized essentially by the equality of the circumferential principal stress with either one of the two principal stresses in an axial plane, namely either $\sigma_\theta = \sigma_2$ or $\sigma_\theta = \sigma_1$ according as plastic régime A or F applies (see (2·21)). Thus the Haar & von Kármán (1909) hypothesis is satisfied by the present plastic régimes. In proceeding, it is convenient to characterize the plastic régimes A and F, respectively, by the introduction of a pure number ϖ with values $+1$ and -1.

Consider first the equations governing the stress field. There are available two equilibrium equations (2·1) and two yield conditions (2·21) for the determination of four stress components σ_r, σ_θ, σ_z and τ_{rz}. Therefore the stress field is statically determinate. Now from (2·8, 21), the yield criterion may be expressed as

$$\left.\begin{aligned} \sigma_\theta &= \tfrac{1}{2}(\sigma_r+\sigma_z)-\varpi\{\tfrac{1}{4}(\sigma_r-\sigma_z)^2+\tau_{rz}^2\}^{\frac{1}{2}}, \\ (\sigma_\theta+Y)/k &= \tfrac{1}{2}(\sigma_r+\sigma_z)+\varpi\{\tfrac{1}{4}(\sigma_r-\sigma_z)^2+\tau_{rz}^2\}^{\frac{1}{2}}, \\ \text{where}\qquad k &= \frac{1+\varpi\sin\phi}{1-\varpi\sin\phi}, \quad Y = \frac{2\varpi c\cos\phi}{1-\varpi\sin\phi}. \end{aligned}\right\} \tag{4·3·1}$$

The elimination of σ_θ from (4·3·1) gives

$$c\cos\phi-\tfrac{1}{2}\sin\phi(\sigma_r+\sigma_z) = \{\tfrac{1}{4}(\sigma_r-\sigma_z)^2+\tau_{rz}^2\}^{\frac{1}{2}}. \tag{4·3·2}$$

By itself, the yield criterion imposes just two independent conditions on the four stress components. It follows that possible yield stress states may be represented in terms of two independent parameters which it is convenient here to choose as a pressure

$$P(r,z) = -\tfrac{1}{2}(\sigma_r+\sigma_z) = -\tfrac{1}{2}(\sigma_1+\sigma_2) \tag{4·3·3}$$

and the angle $\eta(r,z)$ which specifies the orientation of the principal axes of stress. Let

$$Q(r,z) = \tfrac{1}{2}(\sigma_1-\sigma_2) \geqslant 0, \tag{4·3·4}$$

and then, from (2·9) and (4·3·1), the stresses are given generally by

$$\left.\begin{aligned} &\sigma_r = -P+Q\cos 2\eta, \quad \sigma_z = -P-Q\cos 2\eta, \quad \sigma_\theta = -P-\varpi Q, \quad \tau_{rz} = Q\sin 2\eta, \\ &\text{where now}\qquad Q = P\sin\phi+c\cos\phi > 0. \end{aligned}\right\} \tag{4·3·5}$$

If $Q = 0$, then there is an isotropic stress state with $\sigma_r = \sigma_z = \sigma_\theta = c\cot\phi$ and $\tau_{rz} = 0$, which corresponds of course to the vertex of the yield pyramid. However, this situation has explicitly been excluded. Admissible yield stress states must satisfy the condition of equilibrium. Thus, the functions P and η satisfy two simultaneous first-order non-linear partial differential equations found from the substitution of (4·3·5) into (2·1), namely

$$\left.\begin{aligned} (1-\sin\phi\cos 2\eta)\frac{\partial P}{\partial r}-\sin\phi\sin 2\eta\frac{\partial P}{\partial z}+2Q\left\{\sin 2\eta\frac{\partial \eta}{\partial r}-\cos 2\eta\frac{\partial \eta}{\partial z}-\frac{1}{2r}(\varpi+\cos 2\eta)\right\} &= 0, \\ \sin\phi\sin 2\eta\frac{\partial P}{\partial r}-(1+\sin\phi\cos 2\eta)\frac{\partial P}{\partial z}+2Q\left\{\cos 2\eta\frac{\partial \eta}{\partial r}+\sin 2\eta\frac{\partial \eta}{\partial z}+\frac{1}{2r}\sin 2\eta\right\}+\rho g &= 0. \end{aligned}\right\} \tag{4·3·6}$$

The discussion of the characteristics of the stress equations (4·3·6) is quite straightforward. In the usual way, if P and η are given on Γ, then $\partial P/\partial r$, $\partial P/\partial z$, $\partial \eta/\partial r$ and $\partial \eta/\partial z$ on Γ are given from (4·3·6) together with

$$\left.\begin{aligned} \frac{\partial P}{\partial r}\mathrm{d}r+\frac{\partial P}{\partial z}\mathrm{d}z-\mathrm{d}P &= 0, \\ \frac{\partial \eta}{\partial r}\mathrm{d}r+\frac{\partial \eta}{\partial z}\mathrm{d}z-\mathrm{d}\eta &= 0. \end{aligned}\right\} \tag{4·3·7}$$

Then discontinuities in the first-order derivatives of P and η across Γ only occur if the matrix of the coefficients in (4·3·6, 7) has rank 3, so that all fourth-order determinants formed from this matrix vanish. The characteristic directions $\mathrm{d}z/\mathrm{d}r$ are given from the vanishing of the determinant formed from the elements of the first four columns of this matrix, i.e.

$$(\sin\phi+\cos 2\eta)\,(\mathrm{d}r)^2+2\sin 2\eta\,\mathrm{d}r\,\mathrm{d}z+(\sin\phi-\cos 2\eta)\,(\mathrm{d}z)^2=0, \tag{4·3·8}$$

and therefore

$$\mathrm{d}z/\mathrm{d}r=\tan(\eta-\tfrac{1}{4}\pi-\tfrac{1}{2}\phi),\quad \tan(\eta+\tfrac{1}{4}\pi+\tfrac{1}{2}\phi). \tag{4·3·9}$$

Accordingly, the characteristics are real and distinct, and therefore the stress equations are hyperbolic. The two characteristics through any point intersect at a constant angle $\frac{1}{2}\pi+\phi$, so that in general the characteristics are not orthogonal.

Now let

$$\psi=\eta-(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi), \tag{4·3·10}$$

and name the characteristic with slope $\tan\psi$ an α line and that with slope $\tan(\psi+\frac{1}{2}\pi+\phi)$ a β line. Then the characteristic relations, which result from the vanishing of all other fourth-order determinants formed from the above matrix, are found in the form

$$\left.\begin{aligned}\cos\phi\,\mathrm{d}P+2Q\,\mathrm{d}\psi-\frac{\varpi Q}{r}\{\cos(\psi+\phi)-\varpi\sin\psi\}\,\mathrm{d}s_\alpha-\rho g\sin(\psi+\phi)\,\mathrm{d}s_\alpha=0\quad&\text{on an }\alpha\text{ line,}\\ \cos\phi\,\mathrm{d}P-2Q\,\mathrm{d}\psi-\frac{Q}{r}\{\cos(\psi+\phi)-\varpi\sin\psi\}\,\mathrm{d}s_\beta-\rho g\cos\psi\,\mathrm{d}s_\beta=0\quad&\text{on a }\beta\text{ line,}\end{aligned}\right\} \tag{4·3·11}$$

where s_α and s_β are arc lengths measured along the α and β lines, respectively.

Consider next the equations governing the velocity field. First, there is the equation of isotropy

$$\dot\epsilon_r-\dot\epsilon_z=2\dot\gamma_{rz}\cot 2\eta, \tag{4·3·12}$$

where η is known from the solution of the stress equations. Secondly, there is the equation which arises from the elimination of the parameters $\dot\lambda$ and μ from the flow rule (2·21), namely

$$\dot\epsilon_r+\dot\epsilon_z+(1+\varpi\sin\phi)\,\dot\epsilon_\theta=\sin\phi\{(\dot\epsilon_r-\dot\epsilon_z)^2+4\dot\gamma_{rz}^2\}^{\frac{1}{2}}. \tag{4·3·13}$$

Substituting from (4·3·12) into (4·3·13) and making use of the fact that $\dot\epsilon_r-\dot\epsilon_z$ and $\cos 2\eta$ have the same sign (see (2·13)), it is found that (4·3·13) is simply

$$\dot\epsilon_r+\dot\epsilon_z+(1+\varpi\sin\phi)\,\dot\epsilon_\theta=(\dot\epsilon_r-\dot\epsilon_z)\sin\phi\sec 2\eta. \tag{4·3·14}$$

Accordingly, the velocity components u and w satisfy the two simultaneous partial differential equations

$$\left.\begin{aligned}\sin 2\eta\frac{\partial u}{\partial r}-\cos 2\eta\frac{\partial u}{\partial z}-\cos 2\eta\frac{\partial w}{\partial r}-\sin 2\eta\frac{\partial w}{\partial z}&=0,\\ (\sin\phi-\cos 2\eta)\frac{\partial u}{\partial r}-(\sin\phi+\cos 2\eta)\frac{\partial w}{\partial z}-(1+\varpi\sin\phi)\cos 2\eta\frac{u}{r}&=0.\end{aligned}\right\} \tag{4·3·15}$$

A discussion similar to that given for the stress equations (4·3·6) shows that these equations are hyperbolic, with the same characteristics (4·3·9) as the stress field. The characteristic relations are, in the same notation as before,

$$\left.\begin{aligned}\cos\psi\,\mathrm{d}u+\sin\psi\,\mathrm{d}w+(1+\varpi\sin\phi)\,u\frac{\mathrm{d}s_\alpha}{2r}=0\quad&\text{on an }\alpha\text{ line,}\\ \sin(\psi+\phi)\,\mathrm{d}u-\cos(\psi+\phi)\,\mathrm{d}w-(1+\varpi\sin\phi)\,u\frac{\mathrm{d}s_\beta}{2r}=0\quad&\text{on a }\beta\text{ line.}\end{aligned}\right\} \tag{4·3·16}$$

The velocity field must also satisfy the inequality conditions of (2·21), namely

$$(\dot{\epsilon}_1, -\dot{\epsilon}_2, -\varpi\dot{\epsilon}_\theta) \geqslant 0, \tag{4·3·17}$$

and it is easily shown, with use of (4·3·13), that necessary and sufficient conditions for (4·3·17) to hold are that

$$\dot{\epsilon}_r + \dot{\epsilon}_z \geqslant -k\dot{\epsilon}_\theta, \quad -\varpi\dot{\epsilon}_\theta \geqslant 0. \tag{4·3·18}$$

Finally, consider the situation of a discontinuity in stress across a curve Γ (see figure 4 (ii)) that separates regions in which either of plastic régimes A and F applies. Let the normal to Γ at P make an angle Ψ with the r direction as shown in figure 4 (ii). Then, with use of (4·3·5), it follows that

$$\left.\begin{aligned} \sigma_n &= -P + Q\cos 2(\eta - \Psi), \\ \sigma_s &= -P - Q\cos 2(\eta - \Psi), \\ \tau_{ns} &= Q\sin 2(\eta - \Psi). \end{aligned}\right\} \tag{4·3·19}$$

Now $[\sigma_n, \tau_{ns}] = 0$ but, by hypothesis, $[\sigma_s] \neq 0$. It is straightforward to show that $\eta^\pm$, defined in (3·1), satisfy the relation

$$\cos(\eta^+ + \eta^- - 2\Psi) = \sin\phi\cos(\eta^+ - \eta^-). \tag{4·3·20}$$

If $[\eta]$ is known, then $[\sigma_s]$ can be found from, say, the condition $[Q\sin 2(\eta - \Psi)] = 0$. Similarly $[\sigma_\theta]$ can be found, and it is interesting to note that only $[\sigma_\theta]$ depends upon $\varpi^\pm$, i.e. upon which of régimes A and F applies in $\mathfrak{D}^-$ and in $\mathfrak{D}^+$. The relation (4·3·20) also occurs in work by Shield (1954*a*) concerning plane plastic strain.

4·4. *Group IV. Plastic régime AF*

The single plastic régime comprising group IV is regular.

Consider first the equations governing the velocity field. The flow rule (2·21) shows that $\dot{\epsilon}_\theta = 0$, and therefore

$$u = 0. \tag{4·4·1}$$

Thus the only non-zero strain-rates are

$$\dot{\epsilon}_z = \frac{\partial w}{\partial z}, \quad \dot{\gamma}_{rz} = \frac{1}{2}\frac{\partial w}{\partial r}. \tag{4·4·2}$$

The elimination of the parameter $\dot{\lambda}$ from the flow rule (2·21) furnishes an equation for w, namely

$$\frac{\partial w}{\partial z} = \sin\phi\left\{\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\}^{\frac{1}{2}} \geqslant 0, \tag{4·4·3}$$

no further inequality conditions on w being necessary. Thus, w satisfies the equation

$$\left.\begin{aligned} \varpi\sin\phi\frac{\partial w}{\partial r} + \cos\phi\frac{\partial w}{\partial z} &= 0, \\ \text{where} \qquad \varpi &= \operatorname{sgn}\left(-\frac{\partial w}{\partial r}\right). \end{aligned}\right\} \tag{4·4·4}$$

Now introduce rectangular Cartesian co-ordinates X, Y defined by

$$X = r\cos\phi - \varpi z\sin\phi, \quad Y = \varpi r\sin\phi + z\cos\phi. \tag{4·4·5}$$

Then (4·4·4) shows that $\partial w/\partial Y = 0$, so that

$$w = w(X) = w(r\cos\phi - \varpi z\sin\phi). \tag{4·4·6}$$

Also, from (2·13) and (4·4·3, 4),

$$\tan 2\eta = \varpi\cot\phi, \quad \cos 2\eta \leqslant 0, \tag{4·4·7}$$

so that

$$\eta = \tfrac{1}{2}\pi + \varpi(\tfrac{1}{4}\pi - \tfrac{1}{2}\phi). \tag{4·4·8}$$

Now consider the equations governing the stress field. There are available the two equilibrium equations (2·1), the isotropy equation (2·14) and the yield condition (2·21) for the determination of the four stress components σ_r, σ_θ, σ_z and τ_{rz}. The equation of isotropy shows that

$$\varpi\tau_{rz} = \tfrac{1}{2}\cot\phi(\sigma_r - \sigma_z). \tag{4·4·9}$$

From (2·8, 10, 17) and (4·4·7, 9), the yield criterion (2·21) may be written in the form

$$\cos^2\phi\,\sigma_r - (1+\sin^2\phi)\,\sigma_z + 2c\cos\phi\,\sin\phi = 0. \tag{4·4·10}$$

The elimination of σ_r and τ_{rz} between (2·1) and (4·4·9, 10) gives an equation for σ_z, namely

$$\left(\varpi\sin\phi\frac{\partial}{\partial r} + \cos\phi\frac{\partial}{\partial z}\right)\{r(\sigma_z - c\cot\phi + \tfrac{1}{2}\varpi\rho gr\cot\phi)\} = 0. \tag{4·4·11}$$

Thus, it follows that

$$r(\sigma_z - c\cot\phi + \tfrac{1}{2}\varpi\rho gr\cot\phi) = F(X) = F(r\cos\phi - \varpi z\sin\phi), \tag{4·4·12}$$

where F is an arbitrary function of X only. Thus, (2·1) and (4·4·9, 10, 12) now determine all the stress components in the form

$$\left.\begin{aligned}\sigma_r &= c\cot\phi - \varpi\frac{1+\sin^2\phi}{\sin 2\phi}\rho gr + \frac{1+\sin^2\phi}{\cos^2\phi}\frac{F}{r},\\ \sigma_\theta &= c\cot\phi - 2\varpi\frac{1+\sin^2\phi}{\sin 2\phi}\rho gr + \sec\phi\frac{\mathrm{d}F}{\mathrm{d}X},\\ \sigma_z &= c\cot\phi - \tfrac{1}{2}\varpi\cot\phi\,\rho gr + \frac{F}{r},\\ \tau_{rz} &= -\tfrac{1}{2}\rho gr + \varpi\tan\phi\frac{F}{r},\end{aligned}\right\}\quad(\phi > 0). \tag{4·4·13}$$

Finally, certain inequality conditions on the stresses must be satisfied. The condition $\sigma_z \geqslant \sigma_r$ (see (2·9) and (4·4·7)) requires that

$$F - \tfrac{1}{2}\varpi\rho gr^2\cot\phi \leqslant 0, \quad (\phi > 0). \tag{4·4·14}$$

Also, the condition $\sigma_1 \geqslant \sigma_\theta \geqslant \sigma_2$ (see (2·21)) requires that

$$\begin{aligned}\frac{1-\sin\phi}{\cos\phi}(F - \tfrac{1}{2}\varpi\rho gr^2\cot\phi) &\geqslant r\frac{\mathrm{d}F}{\mathrm{d}X} - \varpi\frac{1+\sin^2\phi}{\sin\phi}\rho gr^2\\ &\geqslant \frac{1+\sin\phi}{\cos\phi}(F - \tfrac{1}{2}\varpi\rho gr^2\cot\phi), \quad (\phi > 0).\end{aligned} \tag{4·4·15}$$

The condition (4·4·14) is implied by (4·4·15). The above formal solution for the stress components needs modification in the case $\phi = 0$. The procedure is to replace $F(X)$ by

$\{G(X) - c\operatorname{cosec}\phi + \frac{1}{2}\varpi\rho g X^2\cot\phi\}$, and to let $\phi \to 0$. Then it is found that expressions for the stress components are

$$\left.\begin{aligned} \sigma_r &= \varpi c\frac{z}{r} - \rho g z + \frac{G}{r}, \\ \sigma_\theta &= \qquad\; -\rho g z + \frac{\mathrm{d}G}{\mathrm{d}r}, \\ \sigma_z &= \varpi c\frac{z}{r} - \rho g z + \frac{G}{r}, \\ \tau_{rz} &= -\varpi c, \end{aligned}\right\} \quad (\phi = 0), \tag{4·4·16}$$

where $G = G(r)$. It is to be noted that now $\sigma_r = \sigma_z$, and the condition that σ_θ is the intermediate stress is

$$\varpi c\frac{z}{r} + c \geqslant \frac{\mathrm{d}G}{\mathrm{d}r} - \frac{G}{r} \geqslant \varpi c\frac{z}{r} - c, \quad (\phi = 0). \tag{4·4·17}$$

Again, if the effect of soil weight is negligible, then (4·4·15) simplifies to

$$\tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\phi\right)F \geqslant r\frac{\mathrm{d}F}{\mathrm{d}X} \geqslant \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right)F, \quad (\phi > 0,\ \rho g L/c \ll 1). \tag{4·4·18}$$

4·5. *Summary of structure of field equations*

In general, it is necessary perforce to adopt an heuristic method of procedure in the solution of plastic problems. This situation is due, of course, to the requirement that the plastic régimes involved, together with the extent of the regions in which they occur, must be postulated *a priori*. Now, naturally, this procedure is planned, at least tentatively, on the basis of physical intuition and past experience, taken in conjunction with the nature of the boundary conditions appropriate to the problem in hand, and with knowledge of the fundamental nature of the stress and velocity fields generated by the various possible plastic régimes. It is appropriate, therefore, to summarize briefly here the salient features of the structure of the field equations for the plastic régimes as found in §§ 4·1–4.

For convenience of discussion, the terms *statically determinate* and *kinematically determinate* are used in their *loose sense*. Thus, a stress field is said to be statically determinate if there are available for its determination as many equations involving only the stresses as there are unknown stress components, no consideration being given to the availability of sufficient stress boundary conditions. Similar remarks apply to kinematically determinate velocity fields.

The principal results that have been found are as follows:

(*a*) *Group I. Plastic régimes B and E.* These régimes are semi-isotropic and singular. In general, no solution is possible for the stress equations. In two special cases, when either the internal friction of a soil, or its weight, are neglected, the stresses are given by simple explicit expressions. The velocity field is indeterminate.

(*b*) *Group II. Plastic régimes AB and EF.* These régimes are regular. The velocity field is kinematically determinate and is hyperbolic, with characteristic directions coincident with the principal strain-rate directions (and, equivalently, the principal stress directions). The stress field is hyperbolic, its characteristic directions coinciding with those of the velocity field.

(*c*) *Group III. Plastic régimes A and F.* These régimes are singular and in accord with the hypothesis of Haar & von Kármán. The stress field is statically determinate and is hyperbolic with characteristic directions inclined at angles $\pm(\frac{1}{4}\pi+\frac{1}{2}\phi)$ with the direction of the algebraically greater of the principal stresses in the axial plane. The velocity field is hyperbolic, its characteristic directions coinciding with those of the stress field.

(*d*) *Group IV. Plastic régime AF.* This régime is regular. The directions of principal strain-rate (and, equivalently, of principal stress) are fixed in direction, and the stress and velocity equations are therefore virtually uncoupled. Thus, the stress field is statically determinate and the velocity field is kinematically determinate. The stress components satisfy simple linear first-order partial differential equations and are given in terms of a single arbitrary function. The radial velocity component is zero; the axial velocity component, which satisfies a simple linear first-order partial differential equation with constant coefficients, is given in terms of a second arbitrary function.

The most striking feature of the structure of the field equations for the various plastic régimes is that in no case are the equations elliptic. In problems whose solution involves the determination of the position of internal boundaries, as is normally to be envisaged, the advantage of dealing with non-elliptic partial differential equations is obvious.

The relative importance of the stress and velocity fields generated by the various plastic régimes can only become apparent in the course of the solution of problems. However, it is possible to make some preliminary conjectures, some of which have been anticipated at the beginning of this section. First, except under very special conditions, no solution is possible to the field equations for group I. Secondly, the fact that for group IV the radial velocity component is zero suggests that its application will be very limited. Accordingly, interest is now centred on the plastic régimes of groups II and III which all generate hyperbolic stress and velocity fields with coincident characteristics. The latter case is the simpler of the two, because the stress fields are statically determinate. The above arguments seemingly focus attention in preliminary investigations particularly (although not exclusively) on the stress and velocity fields generated by the plastic régimes of group III in connexion with the solution of problems of interest. This conclusion lends some support to, although of course it clearly does not substantiate, the *ad hoc* adoption of the Haar & von Kármán régimes often made by writers (see, for example, Berezancev 1955 and Ishlinskiĭ 1944) in the solution of problems. Hill (1950) has also commented on this procedure. In part III of this paper, attention is confined to some illustrative problems of fundamental interest whose solution does involve attention only to these particular fields.

PART III. APPLICATIONS OF THEORETICAL ANALYSIS

5. Incipient plastic flow in a right circular cylindrical sample of soil under uni-axial compressive stress

In this section, a discussion will be given of the incipient plastic flow of a right circular cylindrical sample of soil stressed to the yield-point by uni-axial compressive stresses parallel to its axis. The purpose of this discussion is mainly illustrative, but the analysis is also of interest in connexion with the compression testing of small soil samples. Haythornthwaite (1960) has given a closely related discussion directly pertaining to the theory of the tri-axial test for soils, failure being due to either axial extension or axial compression, particularly with reference to sands in the former case.

Take the z axis to coincide with the axis of the sample, supposed vertical, and then let the sample occupy the region,

$$0 \leqslant r \leqslant R, \quad 0 \leqslant z \leqslant H, \tag{5·1}$$

R and H being its radius and height (see figure 5).

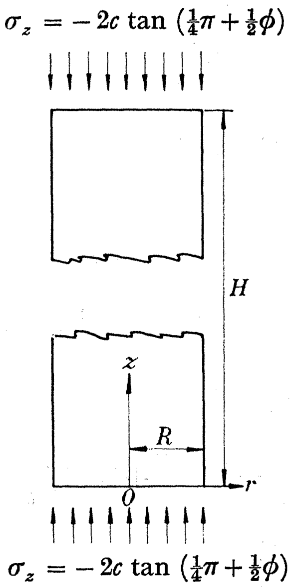

FIGURE 5. Circular cylindrical sample of soil stressed to yield-point under uni-axial compression.

The equilibrium equations (2·1) are identically satisfied if

$$\sigma_z = (\sigma_z)_{z=0} - \rho g z, \tag{5·2}$$

all other stress components being zero. As here $\sigma_1 = 0$, $\sigma_2 \leqslant 0$ and $\sigma_3 = 0$, it follows that plastic régime F applies at the onset of plastic yielding. If gravity effects are important, it is clear that the restriction

$$-(\sigma_z)_{z=H} = -(\sigma_z)_{z=0} + \rho g H \leqslant 2c \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi) \tag{5·3}$$

must apply. Inequality in (5·3) means that yielding nowhere occurs, whereas equality in (5·3) means that plastic flow is initiated just at the section $z = H$. Thus, plastic flow, when it occurs, is simply restricted to the lower end of the sample. However, the situation when gravity effects are negligible, say $\rho g H/c \ll 1$, is very much more interesting because plastic yielding can now occur throughout regions of the sample, and this is possible when

$$\sigma_z = \text{const.} = -2c \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi), \tag{5·4}$$

all other stress components of course vanishing. The stress field is manifestly quite trivial but, nevertheless, surprisingly interesting results are obtained in connexion with the velocity field. In the present circumstances of incomplete specification of the velocity boundary conditions, it is not expected that the mode of deformation in incipient plastic flow is necessarily unique. In fact, in this section, several different velocity fields will be derived as illustrative examples of possible solutions. The question of uniqueness in the theory of a rigid-plastic solid has been discussed in detail by Hill (1956*a*, *b*, 1957*a*, *b*). Although the general analysis of § 4·3, with $\varpi = -1$, applies to the present problem, it is more convenient here to make a direct approach.

The boundary condition that the normal velocity component is constant over an end of the sample will be assumed, namely

$$w = w_0 = \text{const.} > 0 \quad \text{on} \quad 0 \leqslant r \leqslant R, \quad z = 0. \tag{5·5}$$

Now remembering the stress distribution, it follows from (4·3·5) that $1-\cos 2\eta = \sin 2\eta = 0$, and therefore $\eta = 0$. Hence the two families of characteristics comprise parallel straight lines defined by

$$\psi = -(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi) = \text{const.} \quad (< 0). \tag{5·6}$$

In other words, the α and β lines are inclined at constant angles $\mp(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi)$ to the r direction. It may be noted that

$$Q = -c \tan \psi. \tag{5·7}$$

The velocity components u, w must of course satisfy (4·3·16), but because of the relative simplicity of the present problem, it is preferable to consider directly the original velocity equations (4·3·15) which here take the form,

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = \frac{\partial u}{\partial r} + \frac{u}{r} + \tan^2 \psi \frac{\partial w}{\partial z} = 0. \tag{5·8}$$

(*a*) *First velocity field.* The simplest possible type of velocity field is derived on the basis of the plausible assumption that

$$w/w_0 = 1 - z/h, \tag{5·9}$$

where h is an arbitrary positive quantity having the dimensions of length. The case when $h = H$ is just one obvious possibility. Thus, $w = w_0$ at $z = 0$, in agreement with (5·5), and $w = 0$ at $z = h$. The substitution of (5·9) into (5·8) shows first that $u = u(r)$ only and then that

$$u/w_0 = \tan^2 \psi \, r/2h, \tag{5·10}$$

infinite values of u at $r = 0$ being excluded. The velocity field (5·9, 10) satisfies the conditions (4·3·18).

(*b*) *Second velocity field.* Suppose next that plastic flow is confined to the finite region OAB, shown in figure 6, which lies between $z = 0$ and the β line through O. It is supposed here, of course, that $H > R \tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi)$ so that the velocity field can be accommodated. The angle between OA and OB is of course $\psi + \tfrac{1}{2}\pi + \phi = \tfrac{1}{4}\pi + \tfrac{1}{2}\phi = -\psi$. The region above OB is supposed to be rigid.

Two possibilities now arise, according as whether or not the velocity is assumed continuous across OB. In the former case, the appropriate velocity boundary conditions are $u = w = 0$ on OB. In the latter case, the final discussion given in § 3 shows that $[v_n] = [v_s] \tan \phi$, which implies that

$$u = w \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi) \quad \text{on} \quad OB. \tag{5·11}$$

As OB is a β line, (4·3·16) shows that

$$\mathrm{d}u + \tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi)\,\mathrm{d}w + u\frac{\mathrm{d}r}{r} = 0 \quad \text{on} \quad OB. \tag{5·12}$$

The general solution of (5·11, 12) is

$$u = w\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi\right) = Dr^{-\frac{1}{2}} \quad \text{on} \quad OB, \tag{5·13}$$

where D is an arbitrary constant. However, the velocity components on OB are not finite at $r=0$ unless $D=0$, in which case the velocity is not in fact discontinuous across OB. Therefore, the velocity boundary conditions must be taken as

$$u = w = 0 \quad \text{on} \quad OB, \tag{5·14}$$

i.e. the velocity is continuous. Now let

$$u/w_0 = u', \quad w/w_0 = w', \quad r/L = r', \quad z/L = z', \tag{5·15}$$

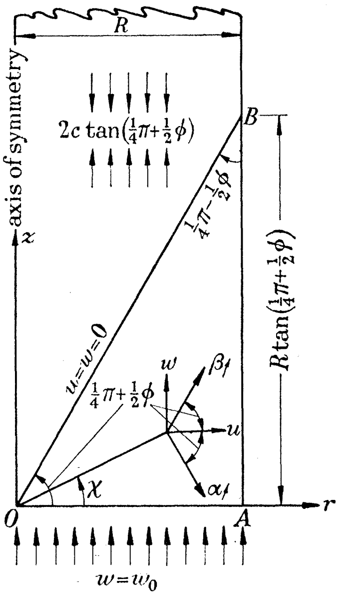

FIGURE 6. Circular cylindrical sample of soil. Second velocity field. Plastic yielding confined to region OAB.

where L is a typical length, so that u', w' and r', z' are non-dimensional velocity components and co-ordinates. Then the problem is to find the solution of

$$\frac{\partial u'}{\partial z'}+\frac{\partial w'}{\partial r'} = \frac{\partial u'}{\partial r'}+\frac{u'}{r'}+\tan^2\psi\frac{\partial w'}{\partial z'} = 0 \tag{5·16}$$

subject to the boundary conditions

$$w' = 1 \quad \text{on} \quad OA \quad \text{and} \quad u' = w' = 0 \quad \text{on} \quad OB. \tag{5·17}$$

The boundary conditions on AB are not specified, and as (5·16, 17) do not involve any fundamental lengths it follows that u' and w' are simply functions of z'/r'. In proceeding, it may therefore be assumed that the velocity components u and w at a general point $P(r, z)$ of the region OAB depend only on the angle χ between OP and the r direction, i.e.

$$\tan\chi = z/r \quad (0 \leqslant \chi \leqslant \tfrac{1}{2}\pi). \tag{5·18}$$

Equations (5·8) may now be written in the form,

$$\left.\begin{aligned} \cos\chi\frac{du}{d\chi}-\sin\chi\frac{dw}{d\chi} &= 0,\\ \sin\chi\frac{du}{d\chi}-u\sec\chi-\tan^2\psi\cos\psi\frac{dw}{d\chi} &= 0,\end{aligned}\right\} \quad (0 \leqslant \chi \leqslant -\psi = \tfrac{1}{4}\pi+\tfrac{1}{2}\phi), \tag{5·19}$$

where u and w are regarded as functions of χ. The boundary conditions are now

$$w/w_0 = 1 \text{ on } \chi = 0 \quad \text{and} \quad u/w_0 = w/w_0 = 0 \text{ on } \chi = -\psi = \tfrac{1}{4}\pi + \tfrac{1}{2}\phi. \tag{5·20}$$

It is straightforward to show that the velocity field is

$$\left.\begin{aligned} u/w_0 &= \frac{2}{\pi}(\tan^2\psi - \tan^2\chi)^{\frac{1}{2}}, \\ w/w_0 &= \frac{2}{\pi}\arccos\,(-\tan\chi/\tan\psi), \end{aligned}\right\} \quad (0 \leqslant \chi \leqslant -\psi), \tag{5·21}$$

the value of the inverse cosine lying in the range $(0, \frac{1}{2}\pi)$ (cf. Haythornthwaite 1960). It is simple to show that (5·21) satisfy the conditions (4·3·18). The functions $u(\chi)$ and $w(\chi)$ are exhibited in figures 7 and 8, respectively, for $\phi = 0°, 20°, 40°$. The resulting deformation of a square grid when the above velocity field is maintained for a short period of time is shown in figure 9 for $\phi = 0°, 20°, 40°$. It should be noted that the velocity field is not single-valued at O. In fact, u/w_0 ranges from 0 to $-\frac{1}{2}\pi\tan\psi$ and w/w_0 ranges from 0 to 1 as χ ranges from 0 to $-\psi$.

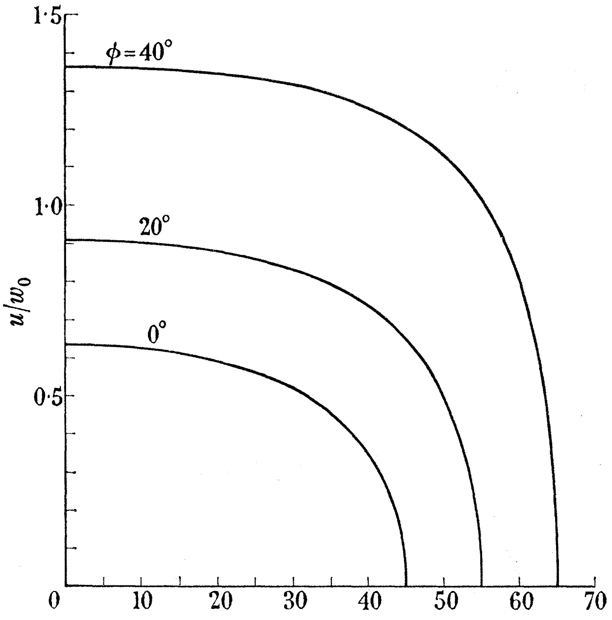

FIGURE 7

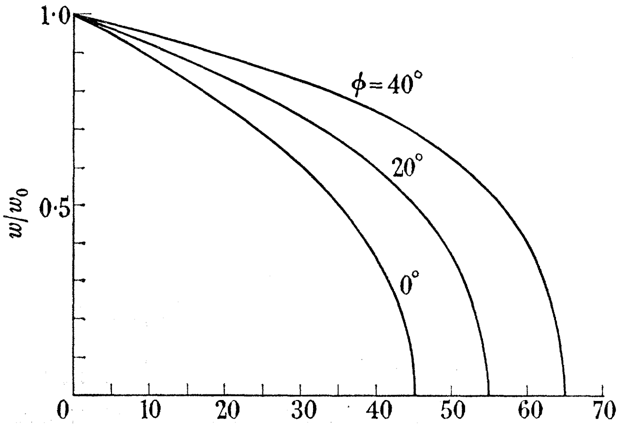

FIGURE 8

FIGURE 7. Variation of $u(\chi)/w_0$ with χ $(0 \leqslant \chi \leqslant \frac{1}{4}\pi + \frac{1}{2}\phi)$ for $\phi = 0°, 20°, 40°$.

FIGURE 8. Variation of $w(\chi)/w_0$ with χ $(0 \leqslant \chi \leqslant \frac{1}{4}\pi + \frac{1}{2}\phi)$ for $\phi = 0°, 20°, 40°$.

(c) *Third velocity field.* Finally, a slightly more complicated form of velocity field will be considered. It is supposed that plastic flow is confined to the finite region $OABC$ of figure 10, where CA and CB are the α and β lines, respectively, passing through the point C on the z axis and at a distance $h = -R\tan\psi$ from O. The boundary conditions on the velocity field are

$$w/w_0 = 1 \text{ on } OA \quad \text{and} \quad u/w_0 = w/w_0 = 0 \text{ on } CB, \tag{5·22}$$

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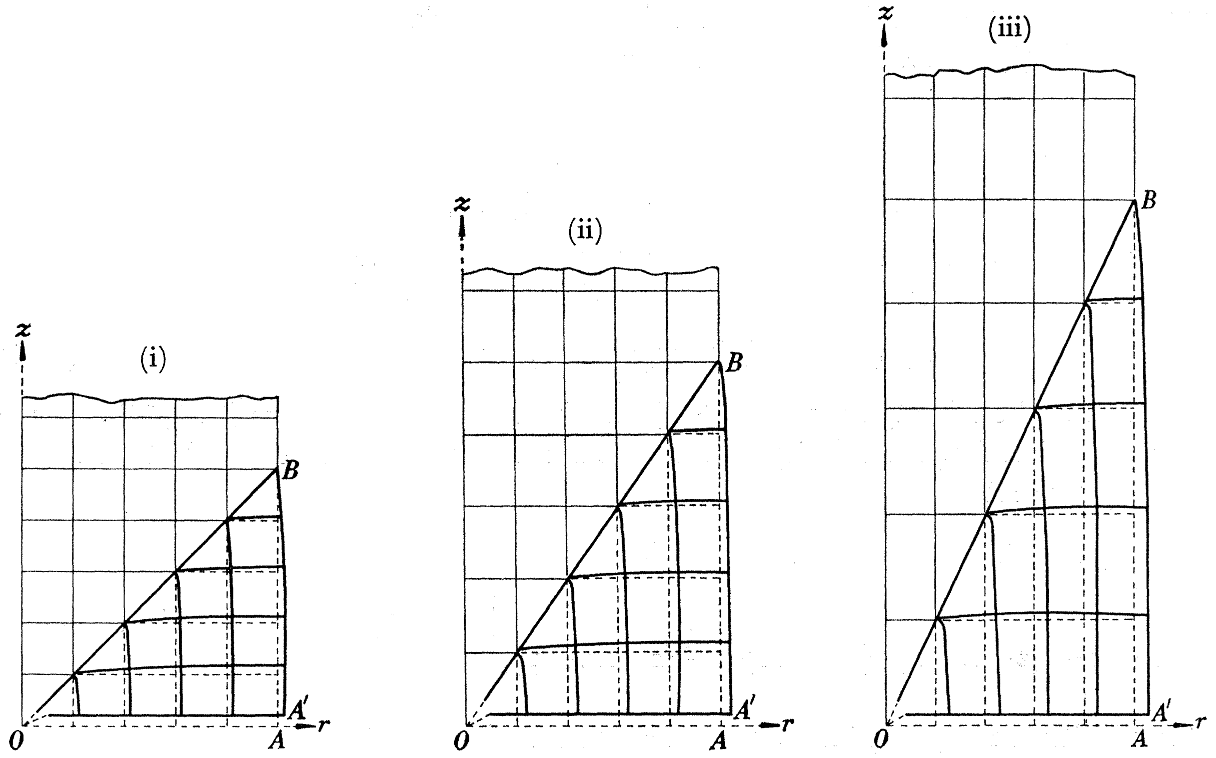

FIGURE 9. Circular cylindrical sample of soil stressed to yield-point in compression. Resulting deformation of a square grid if velocity field of figure 6 is maintained for a short period of time. (i) $\phi = 0°$ (cf. Shield 1955*b*, figure 6). (ii) $\phi = 20°$. (iii) $\phi = 40°$.

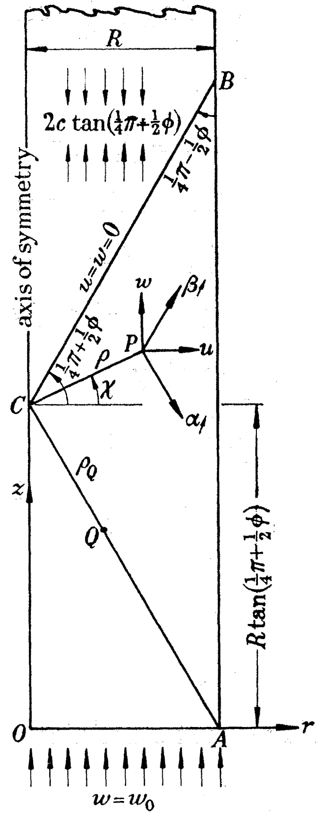

FIGURE 10. Circular cylindrical sample of soil. Third velocity field. Plastic yielding confined to region *OABC*.

the latter conditions following from the fact that the region above *CB* is supposed rigid, exactly as in (*b*) above. The velocity field in the region *OAC* is assumed to be that derived in (*a*) above, now with $h = -R\tan\psi$, namely

$$u/w_0 = -\tan\psi\, r/2R, \quad w/w_0 = 1+\cot\psi\, z/R \quad \text{in} \quad OAC. \tag{5·23}$$

Now as in (*b*) above, the required additional boundary conditions on the velocity field in region *CAB* are that the velocity components u and w should be continuous across *CA*. Thus at a general point Q on *CA*, distant ρ_Q from C (see figure 10) it follows, from (5·23), that

$$u(\rho_Q)/w_0 = -\tfrac{1}{2}\sin\psi\, \rho_Q/R, \quad w(\rho_Q)/w_0 = \cos\psi\, \rho_Q/R. \tag{5·24}$$

The form of the velocity equations (5·16) and the boundary conditions (5·22, 24) suggests the assumption that the velocity components u, w at a general point $P(\rho, \chi)$ of the region *CAB* are of the form,

$$\left.\begin{aligned} &u/w_0 = \rho U(\chi)/R, \qquad w/w_0 = \rho W(\chi)/R,\\ &\text{where}\\ &\rho = \{r^2+(z+R\tan\psi)^2\}^{\frac{1}{2}}, \quad \tan\chi = \frac{1}{r}(z+R\tan\psi), \end{aligned}\right\} \quad (\psi \leqslant \chi \leqslant -\psi = \tfrac{1}{4}\pi + \tfrac{1}{2}\phi). \tag{5·25}$$

The substitution of (5·25) into (5·8) shows that U and W satisfy

$$\left.\begin{aligned} U\sin\chi+\frac{\mathrm{d}U}{\mathrm{d}\chi}\cos\chi+W\cos\chi-\frac{\mathrm{d}W}{\mathrm{d}\chi}\sin\chi = 0,\\ U(\cos\chi+\sec\chi)-\frac{\mathrm{d}U}{\mathrm{d}\chi}\sin\chi+\tan^2\psi\left(W\sin\chi+\frac{\mathrm{d}W}{\mathrm{d}\chi}\cos\chi\right) = 0, \end{aligned}\right\} \quad (\psi \leqslant \chi \leqslant -\psi). \tag{5·26}$$

The boundary conditions on U and W are

$$U = -\tfrac{1}{2}\sin\psi \quad \text{and} \quad W = \cos\psi \quad \text{on} \quad \chi = \psi, \quad \text{and} \quad U = W = 0 \quad \text{on} \quad \chi = -\psi. \tag{5·27}$$

The analysis is now facilitated through the substitutions

$$\left.\begin{aligned} U = \tan\psi\cos\chi F(\zeta), \quad W = \cot\psi\sin\chi G(\zeta),\\ \zeta = \tan\chi\cot\psi, \quad (-1 \leqslant \zeta \leqslant 1). \end{aligned}\right\} \tag{5·28}$$

From (5·26, 28), it is found that F and G satisfy

$$\left.\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}\zeta}-\zeta^2\frac{\mathrm{d}G}{\mathrm{d}\zeta} = 0,\\ 2F-\zeta\frac{\mathrm{d}F}{\mathrm{d}\zeta}+G+\zeta\frac{\mathrm{d}G}{\mathrm{d}\zeta} = 0, \end{aligned}\right\} \quad (-1 \leqslant \zeta \leqslant 1). \tag{5·29}$$

The boundary conditions on F and G are

$$F(1) = -\tfrac{1}{2}, \quad G(1) = 1, \quad F(-1) = 0, \quad G(-1) = 0. \tag{5·30}$$

The elimination of F from (5·29) shows that G satisfies

$$\zeta(1-\zeta^2)\frac{\mathrm{d}^2G}{\mathrm{d}\zeta^2}+(2-\zeta^2)\frac{\mathrm{d}G}{\mathrm{d}\zeta} = 0. \tag{5·31}$$

It is now straightforward to show that the required solution is

$$\left.\begin{aligned} F &= -\frac{1}{2\pi}\{\arccos(-\zeta)+\zeta(1-\zeta^2)^{\frac{1}{2}}\}, \\ G &= \frac{1}{\pi}\{\arccos(-\zeta)+(1-\zeta^2)^{\frac{1}{2}}/\zeta\}, \end{aligned}\right\} \quad (-1 \leqslant \zeta \leqslant 1), \tag{5·32}$$

where the inverse cosine lies in the range $(0, \pi)$. The velocity components are now seen to be given by

$$\left.\begin{aligned} \frac{u/w_0}{\rho/R} &= \frac{1}{2\pi}\cos\chi\{-\omega\tan\psi+\tan\chi\cot\psi(\tan^2\psi-\tan^2\chi)^{\frac{1}{2}}\}, \\ \frac{w/w_0}{\rho/R} &= \frac{1}{\pi}\sin\chi\{\omega\cot\psi-\cot\chi\cot\psi(\tan^2\psi-\tan^2\chi)^{\frac{1}{2}}\}, \\ \text{where} \quad \cos\omega &= -\tan\chi\cot\psi \quad (0 \leqslant \omega \leqslant \pi), \end{aligned}\right\} \tag{5·33}$$

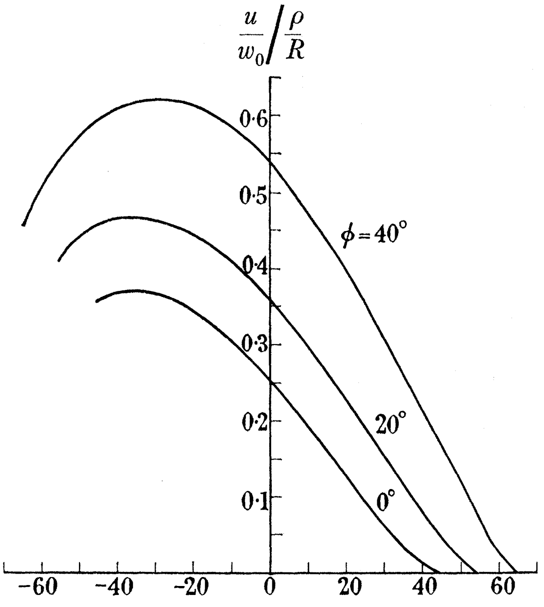

FIGURE 11

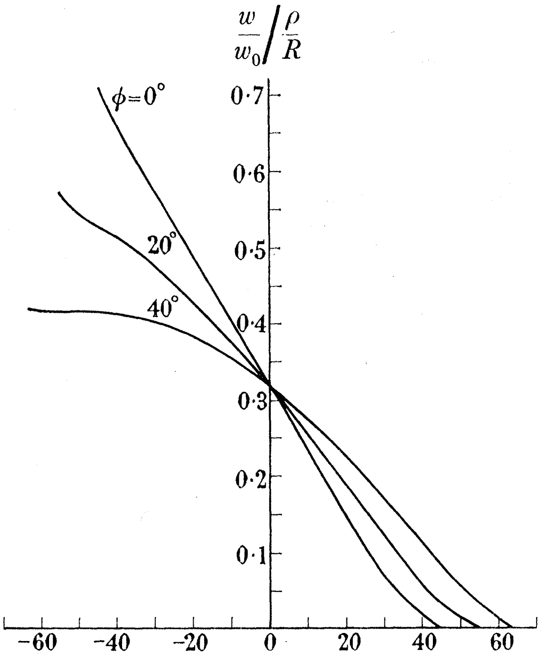

FIGURE 12

FIGURE 11. Variation of $\frac{u}{w_0}\Big/\frac{\rho}{R}$ with χ $(|\chi| \leqslant \frac{1}{4}\pi+\frac{1}{2}\phi)$ for $\phi = 0°, 20°, 40°$.

FIGURE 12. Variation of $\frac{w}{w_0}\Big/\frac{\rho}{R}$ with χ $(|\chi| \leqslant \frac{1}{4}\pi+\frac{1}{2}\phi)$ for $\phi = 0°, 20°, 40°$.

(cf. Haythornthwaite 1960). Finally, it must be shown that (5·33) satisfy the conditions (4·3·18). These conditions, remembering (4·3·1, 5·6), are that

$$\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z} \geqslant -\cot^2\psi\frac{u}{r} \leqslant 0. \tag{5·34}$$

First, the condition $u \geqslant 0$ requires that $F \leqslant 0$, which condition is easily shown to be true. Secondly, the other condition requires with use of (5·8) that

$$\frac{\partial u}{\partial r} \geqslant 0, \tag{5·35}$$

i.e. $\partial(\rho F \cos\chi)/\partial r \leqslant 0$, and this is easily proved to be true. It may be noted that there is no singularity in the velocity field (5·33) at C. The functions u/ρ and w/ρ are exhibited graphically in figures 11 and 12, respectively. The resulting deformation of a square grid when the above velocity field is maintained for a short period of time is shown in figure 13 for $\phi = 0°, 20°, 40°$.

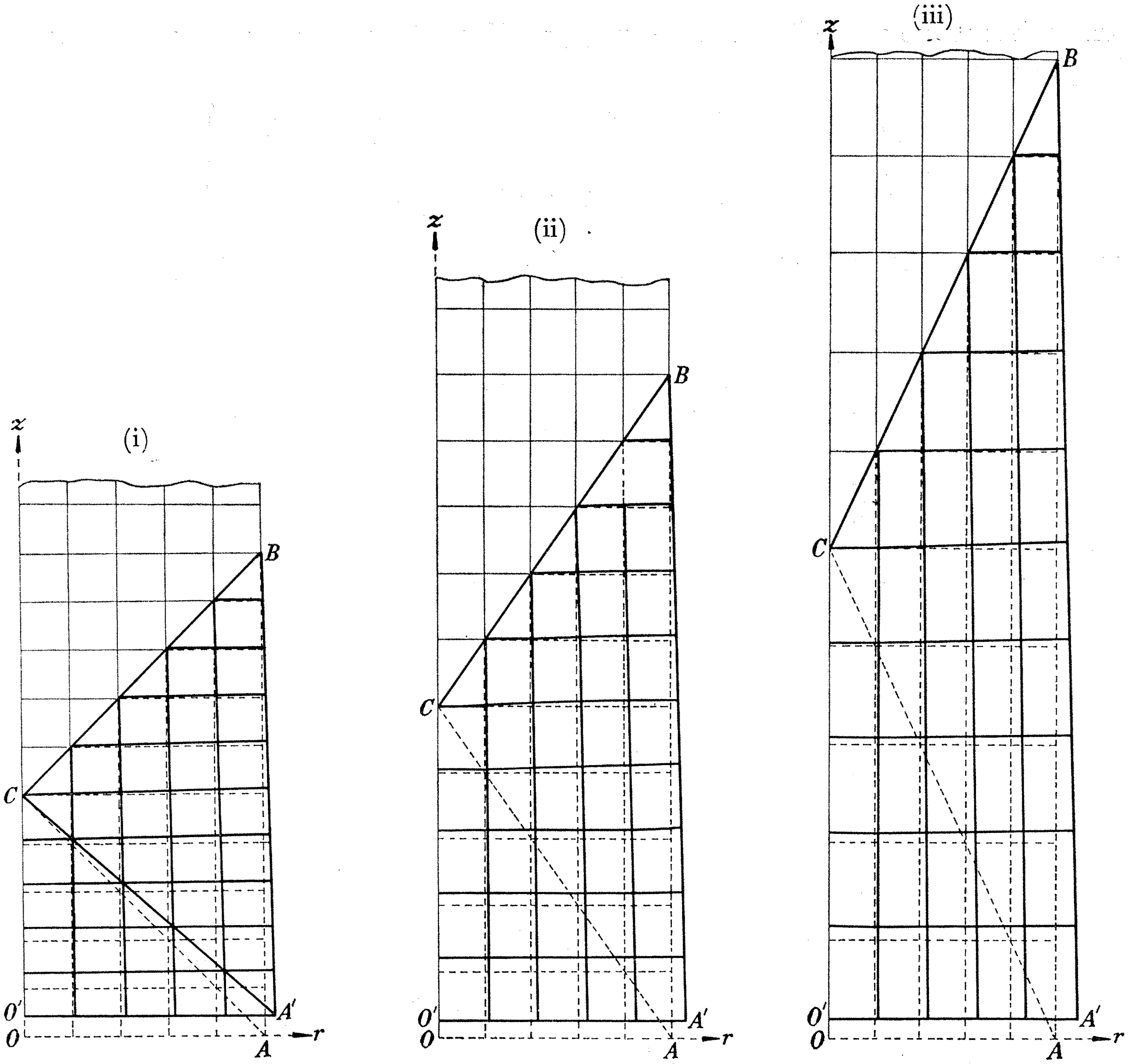

FIGURE 13. Circular cylindrical sample of soil stressed to yield-point in compression. Resulting deformation of a square grid if velocity field of figure 10 is maintained for a short period of time. (i) $\phi = 0°$ (cf. Shield 1955 *b*, figure 7). (ii) $\phi = 20°$. (iii) $\phi = 40°$.

6. INCIPIENT PLASTIC FLOW IN A SEMI-INFINITE REGION OF SOIL DUE TO LOAD APPLIED THROUGH A FLAT-ENDED, SMOOTH, RIGID, CIRCULAR CYLINDER

Let cylindrical polar co-ordinates r, θ, z be defined, the origin being at the centre of the flat end of the cylinder, the radius of which is R (see figure 14). The soil occupies the semi-infinite region $z \geqslant 0$.

The cylinder is taken to be perfectly rigid, with a smooth flat-ended base, and is assumed to be loaded normally. Here, attention is restricted to the case where all effects due to soil weight are neglected, and the accuracy of the results based upon this approximation decreases of course as the parameter $\rho gR/c$ increases. However, the inclusion of effects due to soil weight is straightforward, as has been found by one of the present writers (unpublished work by A. D. C.). Generally it should be noted that in any situation likely to involve extensive regions of soil undergoing plastic deformation, effects due to soil weight are likely to be important and should therefore be included in the analysis.

It is necessary here to take due account of the normal stress corresponding to atmospheric pressure exerted on the surface of natural soil. However, it will be seen later that this situation is equivalent to the simpler one involving a stress-free surface provided that the value of the cohesion of the soil is now suitably augmented by an amount depending upon the value of the atmospheric pressure.

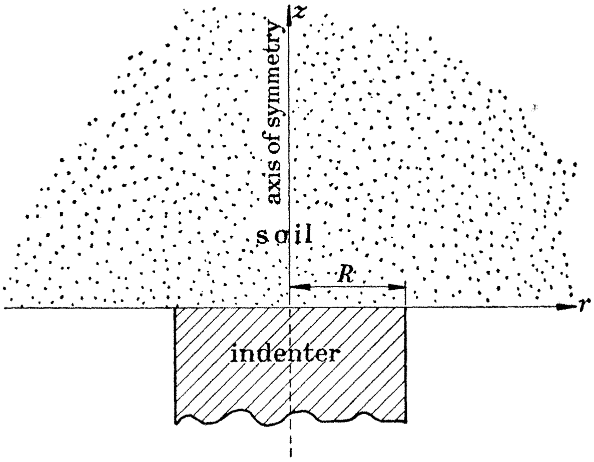

FIGURE 14. Co-ordinate system.

In mathematical terms the above statement of the problem leads to boundary conditions on the stress and velocity components. First, since the punch is smooth and is rigid with a flat end,

$$\left.\begin{aligned} \tau_{rz} &= 0, \\ \text{and} \quad w &= w_0 = \text{const.}, \end{aligned}\right\} \quad \text{on} \quad 0 \leqslant r < R. \tag{6·1}$$

Secondly, since the soil is bounded by a plane surface acted on only by atmospheric pressure, $\tau_{rz} = 0$ and $\sigma_z = -p_a$ on $r > R$, where p_a is the atmospheric pressure. Now, as is easily verified, the equations of equilibrium and also the yield criterion are unaltered by the addition of a (constant) hydrostatic pressure $-p_a$ to the actual stress field provided that at the same time a *relative* cohesion $c^* = c + p_a \tan\phi$ is used in place of the *true* cohesion c. Bearing this in mind, the conditions on the surface may thus be taken as

$$\tau_{rz} = \sigma_z = 0 \quad \text{on} \quad r > R. \tag{6·2}$$

In order to see the possible difference due to the use of c^* in place of c, it may be noted that most soils have values of c (Lb./in.^{2}) and ϕ (deg) in the ranges $0 \leqslant c \leqslant 20$ and $0 \leqslant \phi \leqslant 40$, whereas p_a is approximately 15 Lb./in.^{2}. Thus it is possible for the two terms in c^* to be of comparable magnitude, and for sands, when c is very small and ϕ is about 40°, the second term $p_a \tan\phi$ will be dominant. It should be noted that it is normal in soil mechanics to deal with excess stresses above atmospheric and therefore quoted values of the cohesion of a soil will generally refer to c^*, but care must be taken in any application of the present theory to ascertain that this is the case.

The boundary conditions together with the governing equations (which will be discussed later) suitably define the problem, and it is appropriate at this place to consider the uniqueness of the solution that will be obtained. It should be noted that the velocity boundary conditions are not sufficient completely to determine the velocity field anywhere in the ideal soil. This means that although by arbitrarily imposing further conditions on the solution an acceptable velocity field can be found, it is not to be expected that this solution is unique.

The uniqueness of the corresponding two-dimensional problem, with $\phi = 0$, was discussed by Bishop (1953) and the following arguments are based upon his work. The solution is obtained in three distinct stages. First, once suitable additional boundary conditions have been specified, the velocity components u and w are found throughout the soil, being non-zero only in a certain region near the cylinder and zero elsewhere. Secondly, the stress components are calculated throughout this region. Since the region of plastic deformation must include the whole area of soil in contact with the cylinder, it is clear that the stress components in this region are sufficient to determine the stresses on the cylinder. The solution so far obtained is *incomplete*, and limit analysis only shows that the yield-point loads so obtained are *upper* bounds to the correct values. Thirdly, the stress field is, if possible, extended into the entire rigid region in such a way as to satisfy the conditions of equilibrium and yield. Provided that this can be done in a satisfactory manner, the solution is *complete* and limit analysis shows that the yield-point loads obtained previously are also *lower* bounds to, and hence are identical with, the correct values. Further, a theorem due to Hill (see Bishop 1953) shows that the stress field in the deforming region is unique. However, no conclusions can be drawn as to the uniqueness of the rest of the stress field or of any part of the velocity field. It should be noted that these defects in the nature of the solution are inherent in the method of approach and are in no way a peculiarity of the present problem.

It is necessary to follow an heuristic method when solving the present problem, namely, the relevant régimes are assumed *a priori*, and justified *a posteriori* when a solution is found. The discussion given in § 4 of the consequences of assuming any particular plastic régime suggests that F is appropriate for the present type of problem; this is one of the two Haar & von Kármán régimes which have the property that the intermediate principal stress is equal to one of the other two principal stresses. For this reason it will now be assumed that the equations governing plastic régime F apply. The fact that this plastic régime is statically determinate leads to an inversion of the procedure outlined above, and in any region the stress field is determined before the velocity field, but this does not affect either the argument above or its conclusions.

(*a*) *Incomplete stress field*

(i) *Governing equations*

The governing equations for plastic régime F are, as shown in § 4·3, hyperbolic, and from equations (4·3·9, 10) the characteristics are given by

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \begin{cases} \tan\psi & \text{on an } \alpha \text{ line,} \\ \tan(\psi + \tfrac{1}{2}\pi + \phi) & \text{on a } \beta \text{ line.} \end{cases} \tag{6·3}$$

The relations along these characteristics, with neglect of soil weight and use of the substitution

$$\Lambda = \begin{cases} \cot\phi \ln(Q/c^*) & \text{if} \quad \phi > 0, \\ P/c^* & \text{if} \quad \phi = 0, \end{cases} \tag{6·4}$$

are, from (4·3·11),

$$\left.\begin{aligned} \mathrm{d}\Lambda + 2\mathrm{d}\psi &= -\frac{1}{r}\{\cos(\psi+\phi)+\sin\psi\}\,\mathrm{d}s_\alpha \quad \text{on an } \alpha \text{ line,} \\ \mathrm{d}\Lambda - 2\mathrm{d}\psi &= \frac{1}{r}\{\cos(\psi+\phi)+\sin\psi\}\,\mathrm{d}s_\beta \quad \text{on a } \beta \text{ line.} \end{aligned}\right\} \tag{6·5}$$

In terms of the variables Λ and ψ, the boundary conditions (6·1, 2) are

$$\left.\begin{aligned} &\psi = \tfrac{3}{4}\pi - \tfrac{1}{2}\phi, && \text{on} \quad 0 \leqslant r < R, \\ &\left.\begin{aligned} &\psi = \tfrac{1}{4}\pi - \tfrac{1}{2}\phi, \\ &\Lambda = \cot\phi \ln\cot(\tfrac{1}{4}\pi - \tfrac{1}{2}\phi) \end{aligned}\right\} && \text{on} \quad r > R, \end{aligned}\right\} \quad (z=0). \tag{6·6}$$

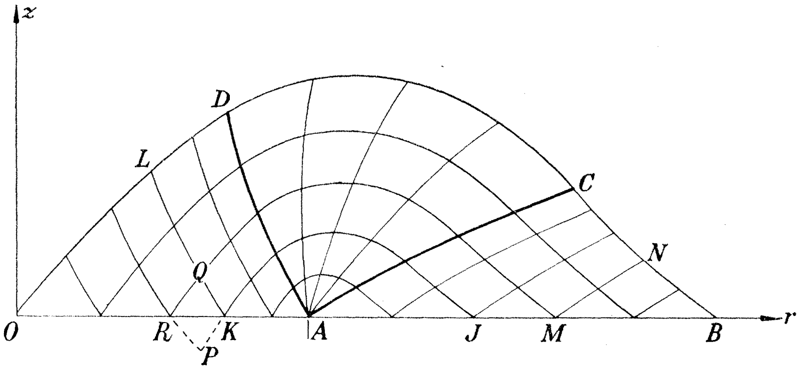

FIGURE 15. Schematic diagram of characteristics net.

Following Shield (1955*b*), the characteristics net is expected to exhibit the geometrical features depicted schematically in figure 15. In this figure the lines such as MN, AC and KL are α lines and those such as MQR are β lines. In the determination of the stress distribution over the base of the circular cylinder, it is only necessary to consider that part of the field bounded by OA, AB and $BCDO$, where $ODCB$ is the β line through O. Now since, from (6·6), both Λ and ψ are known on the line AB (itself not a characteristic), equations (6·5) determine their values in ABC, this being a *second boundary-value problem* in Hill's (1950) nomenclature for the case of plane strain. The values of Λ and ψ on AC, together with the fact that at A there is a singularity in the field (since, for example, ψ is discontinuous at A on OAB), determine the field in the fan ACD, the angle CAD being $\frac{1}{2}\pi$ independently of the angle of friction, ϕ; this is a *first boundary-value problem* (cf. Hill 1950). Finally, the now known values of Λ and ψ on AD together with the value, given by (6·6), of ψ on OA, determine Λ and ψ in ADO, a *third boundary-value or mixed problem* (cf. Hill 1950).

(ii) *Numerical solution*

The numerical procedure is based upon the approximation of the equations (6·5) by finite-difference equations. These finite-difference equations are used, as described below, to determine Λ and ψ at the (initially unknown) point of intersection of an α line and a

β line through two neighbouring points not lying on the same characteristic. Repeated applications of this process determines the entire stress field.

Referring to figure 16, suppose that Λ and ψ are known at P and Q. It is required to find the point of intersection, R, of the α and β lines through P and Q, respectively, together with the values of Λ and ψ at R. For simplicity, suffixes P, Q and R are used to denote values of the variables at the points P, Q and R, respectively.

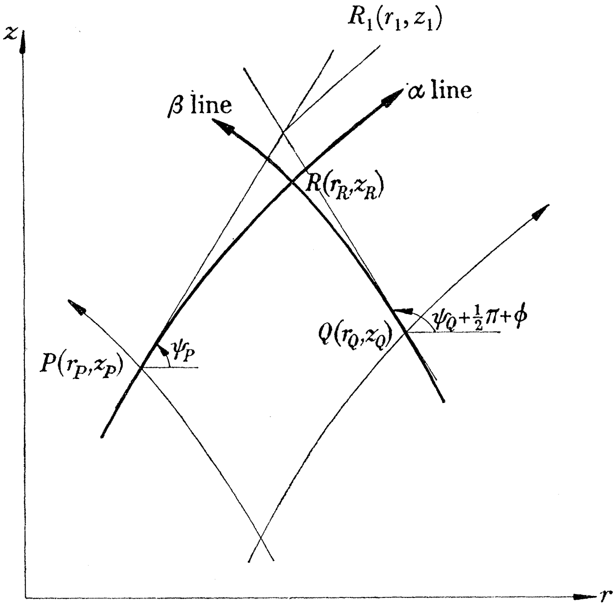

FIGURE 16. Intersection of characteristics through two neighbouring points P and Q of the mesh, showing the first approximation R_1 to the true point of intersection R.

A first approximation to the position of R is taken to be $R_1(r_1, z_1)$, the point of intersection of the tangents to the α and β lines through P and Q respectively. Equations (6·3) show that r_1 and z_1 are given by

$$\left.\begin{aligned} z_1-z_P &= (r_1-r_P)\tan\psi_P, \\ z_1-z_Q &= (r_Q-r_1)\tan(\tfrac{1}{2}\pi-\phi-\psi_Q). \end{aligned}\right\} \tag{6·7}$$

First approximations Λ_1 and ψ_1 to the values of Λ_R and ψ_R are then determined from

$$\left.\begin{aligned} (\Lambda_1-\Lambda_P)+2(\psi_1-\psi_P) &= -2(r_1+r_P)^{-1}\{(r_1-r_P)\cos\phi+(z_1-z_P)\,(1-\sin\phi)\}, \\ (\Lambda_1-\Lambda_Q)-2(\psi_1-\psi_Q) &= 2(r_1+r_Q)^{-1}\{(r_Q-r_1)\cos\phi+(z_1-z_Q)\,(1-\sin\phi)\}. \end{aligned}\right\} \tag{6·8}$$

Second approximations (here taken as final ones) to r_R and z_R are then found from

$$\left.\begin{aligned} z_R-z_P &= (r_R-r_P)\tan\{\tfrac{1}{2}(\psi_P+\psi_1)\}, \\ z_R-r_Q &= (r_Q-r_R)\tan\{\tfrac{1}{2}\pi-\phi-\tfrac{1}{2}(\psi_Q+\psi_1)\}. \end{aligned}\right\} \tag{6·9}$$

Second approximations (here taken as final ones) to Λ_R and ψ_R are determined from equations (6·8) with r_R and z_R in place of r_1 and z_1.

Since the extent of the field, and in particular the position of B (see figure 15), is not known *a priori*, it is more convenient to make calculations along the entire length of a β line, starting from the first one around A, before proceeding to the next one. The first β

line about A depends upon conditions at the singular point A, and special consideration, given later, is required. Suppose, therefore, that the position of the β line JK (see figure 15) and the associated values of Λ and ψ are known. Successive application of the above scheme, starting from M, leads to the determination of the position of MQ and also the associated values of Λ and ψ, but R cannot be determined in this way. This is because the point P, lying on the same α line as R, which would be needed for this determination, lies outside the stress field and is therefore unknown. This apparent lack of data is offset by the fact that z and ψ are already known, i.e. $z = 0$ and $\psi = \frac{3}{4}\pi - \frac{1}{2}\phi$ at R. These extra data are sufficient to determine r and Λ at R, using only the relations (6·3, 5) that pertain to variations along β lines. However, it is more convenient, so far as computation is concerned, to define suitable values at a virtual point P in such a way that, when the above scheme is applied, the conditions on z and ψ at R are automatically satisfied while the values of r and Λ at R are determined. This artifice, discussed by Hill (1950) for the case of plane strain, essentially involves a reflexion of the actual stress field in the plane $z = 0$. In the present problem the appropriate values of the variables at P are

$$r_P = r_Q, \quad z_P = -z_Q, \quad \psi_P = \tfrac{3}{2}\pi - \phi - \psi_Q, \quad \Lambda_P = \Lambda_Q, \qquad (6\cdot10)$$

and it is easily seen that the iterative procedure described above automatically leads to $z_R = 0$, $\psi_R = \frac{3}{4}\pi - \frac{1}{2}\phi$ and at the same time determines r_R and Λ_R.

TABLE 2. NUMERICAL DATA FOR PROBLEM OF INDENTATION OF A PLANE SURFACE BY A CIRCULAR CYLINDER

ϕ (deg)	p_3/c^*	$p_{3,0}/c^*$	p_2/c^*	OB/OA
0	5·69	7·1	5·14	1·58
5	7·44	9·7	6·49	1·71
10	9·98	14	8·34	1·88
15	13·9	22	11·0	2·09
20	20·1	34	14·8	2·37
25	30·5	59	20·7	2·73
30	49·3	110	30·1	3·21
35	85·8	210	46·1	3·89
40	164	430	75·3	4·86

As discussed above, A is a singular point and ψ is multi-valued there. Thus, there is a fan of α lines centred at A. It is convenient to regard the point A as a degenerate β line. Hence A may be regarded as a single infinity of points A_i, all having the same co-ordinates $r_i = R$, $z_i = 0$ but with distinct values of ψ and Λ. Further, since a limiting form, with $\mathrm{d}s_\beta = 0$, of the characteristic relation along a β line (6·5) must hold at A, it follows that if at A_i, $\psi = \frac{1}{4}\pi - \frac{1}{2}\phi + \omega$ $(0 \leqslant \omega \leqslant \frac{1}{2}\pi)$, then $\Lambda = \cot\phi \ln\cot(\frac{1}{4}\pi - \frac{1}{2}\phi) + 2\omega$.

In the detailed calculations the following mesh of characteristics was used. In ABC equally spaced points were taken along AB, and the α and β lines through these points defined the mesh. In ACD the β lines were determined as the continuation of those in ABC; the α lines were chosen by requiring that in passing around the singular point A there were constant increments in the angle ω defined above. In AOD the β lines were again the continuation of those in ACD, while the α lines were taken as those through the points of intersection of the already known β lines and OA.

Table 2 gives values, calculated on the A.R.D.E. digital computer AMOS, for the mean yield-point pressure p_3, the maximum pressure on the cylinder $p_{3,0}$ (which occurs at the origin), and also the ratio OB/OA, all as functions of the angle of internal friction ϕ for $\phi = 0°\ (5°)\ 40°$. Values of all these quantities were computed for successively smaller mesh sizes, and the final values given in table 2 are believed to be accurate to the number of figures

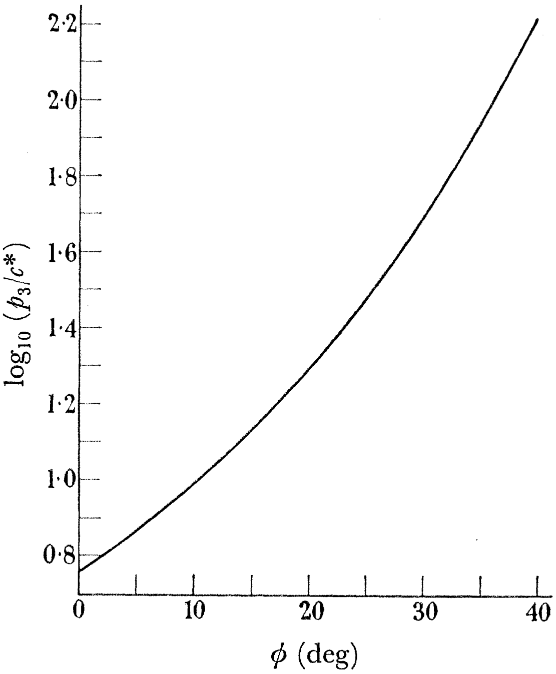

Figure 17. Variation of mean yield-point pressure with angle of internal friction.

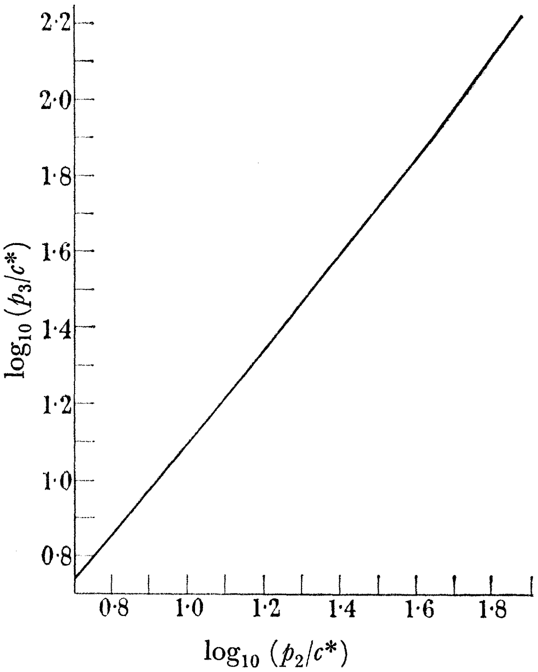

Figure 20. Corresponding axially symmetric and plane strain values of mean yield-point pressure.

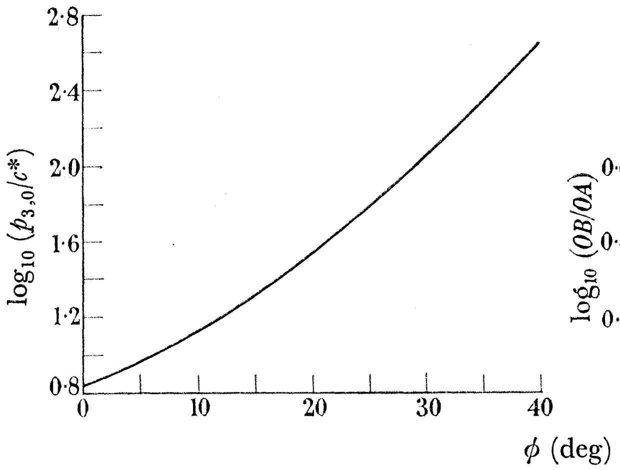

Figure 18. Variation of maximum pressure exerted on the soil with angle of internal friction.

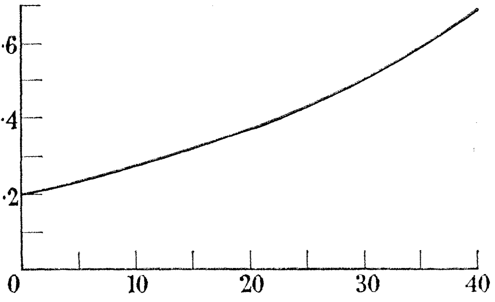

Figure 19. Variation of OB/OA with angle of internal friction.

given. For purposes of comparison, table 2 also shows values of the constant yield-point pressure p_2 for the problem of indentation by a die under plane strain conditions (see Prandtl 1920). In figures 17, 18 and 19 the quantities $p_3, p_{3,0}$ and OB/OA are plotted against ϕ, and in figure 20 the quantity p_3 is plotted against p_2. Figure 20 shows that, as a simple approximation, $\log_{10}p_3$ may be taken as a linear function of $\log_{10}p_2$ over the range of ϕ considered.

Figure 21 shows the calculated net of characteristics, and figure 22 the pressure distribution on the face of the cylinder, for the particular case when $\phi = 20°$. For the other values of ϕ considered here, the characteristics nets exhibit the same geometrical features as for $\phi = 20°$, the major difference being one of scale. This difference in scale is exemplified by the ratio OB/OA which, as shown in figure 19, increases markedly with ϕ.

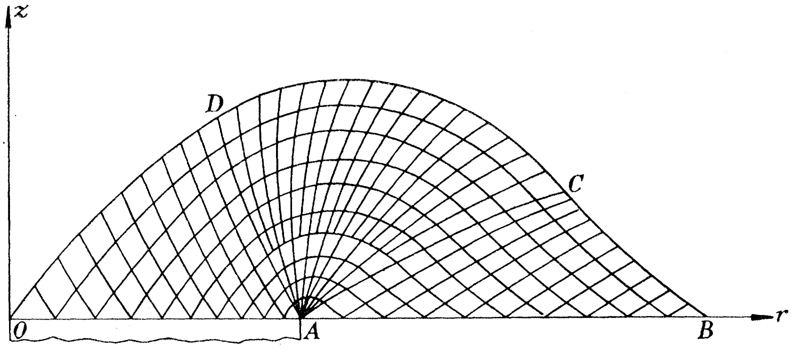

FIGURE 21. Actual characteristics net for $\phi = 20°$.

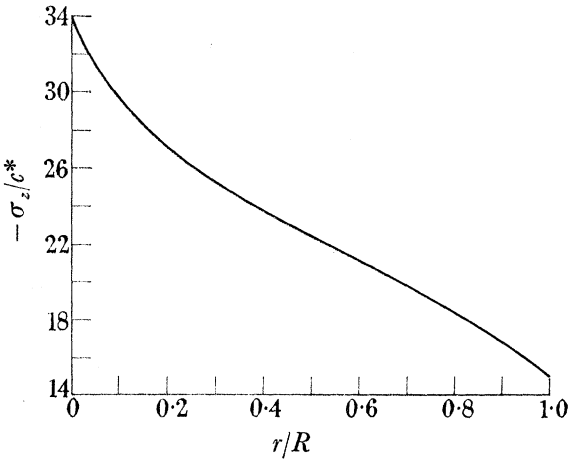

FIGURE 22. Pressure distribution exerted on the soil by the cylinder for $\phi = 20°$.

(b) *Associated velocity field*

(i) *Governing equations*

The equations governing the velocity field are hyperbolic, with the same characteristics as those of the stress equations (see § 4·3). The relations governing the variation of u and w along the characteristics are given by (4·3·16). Here it is convenient to write these relations in terms of the velocity resolutes U and W defined by

$$\left.\begin{aligned} U &= u\cos\psi + w\sin\psi, \\ W &= -u\sin(\psi+\phi) + w\cos(\psi+\phi), \\ \text{or, equivalently,}\qquad u\cos\phi &= U\cos(\psi+\phi) - W\sin\psi, \\ w\cos\phi &= U\sin(\psi+\phi) + W\cos\psi. \end{aligned}\right\} \tag{6·11}$$

In terms of U and W the characteristic relations become

$$\left.\begin{aligned} \mathrm{d}U-\sec\phi(W+U\sin\phi)\,\mathrm{d}\psi &= -\frac{1-\sin\phi}{\cos\phi}\{U\cos(\psi+\phi)-W\sin\psi\}\frac{\mathrm{d}s_\alpha}{2r} \quad \text{on an } \alpha \text{ line,}\\ \mathrm{d}W+\sec\phi(U+W\sin\phi)\,\mathrm{d}\psi &= -\frac{1-\sin\phi}{\cos\phi}\{U\cos(\psi+\phi)-W\sin\psi\}\frac{\mathrm{d}s_\beta}{2r} \quad \text{on a } \beta \text{ line.}\end{aligned}\right\} \quad (6{\cdot}12)$$

The only velocity boundary condition explicitly specified is that $w = w_0$ on OA or, in terms of U and W, remembering (6·6),

$$U-W = 2w_0\cos\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi\right) \quad \text{on} \quad 0 \leqslant r < R, \quad z = 0. \quad (6{\cdot}13)$$

Since this condition by itself is not sufficient to determine the values of U and W, it is necessary, as already mentioned, to assume some further condition. The solution is expected to involve a region of active deformation and a rigid region. Since this must lead to some discontinuities in the velocity components or their spatial derivatives, it follows that in the simplest case the boundary between these two regions must be a characteristic line. It is clear from the geometrical form of the net of characteristics shown in figure 15 that this boundary must be a β line rather than an α line. Further, since the velocity field must accommodate the incipient motion of the cylinder, OA must lie within the region of active deformation. Thus the simplest configuration that can occur is when the β line $ODCB$ is the boundary between the actively deforming region and the rigid region, and attention will now be confined to this case.

Now the region beyond $ODCB$ is rigid, and it can be shown, in a manner similar to that followed in § 5 (*b*) (but now with additional use of (4·3·11)), that

$$U = W = 0 \quad \text{on} \quad ODCB. \quad (6{\cdot}14)$$

The determination of the velocity field is straightforward and is uniformly of the same type (i.e. type (iii) discussed by Hill 1950). The method, which applies throughout the three regions OAD, ACD and ABC, is as follows. Let the β lines be taken in order, with $ODCB$ as the first and the point A as the last. Then the specification of U and W on $ODCB$, together with the relation (6·13) which holds where the second β line meets OA, determines U and W along the entire length of this β line. Repeated application of this method, using the newly determined β line in place of $ODCB$, determines the whole of the field.

It remains to check that the two inequalities (4·3·18) are satisfied. Substituting for u and w in terms of U and W, these conditions reduce to

$$\left.\begin{aligned} &U\cos(\psi+\phi)-W\sin\psi \geqslant 0,\\ &\frac{\partial U}{\partial s_\alpha}+\sin\phi\frac{\partial W}{\partial s_\alpha}-W\cos\phi\frac{\partial\psi}{\partial s_\alpha}+\frac{\partial W}{\partial s_\beta}+\sin\phi\frac{\partial U}{\partial s_\beta}+U\cos\phi\frac{\partial\psi}{\partial s_\beta}\\ &\qquad \geqslant -\frac{1-\sin\phi}{1+\sin\phi}\,\frac{\cos\phi}{r}\{U\cos(\psi+\phi)-W\sin\psi\}.\end{aligned}\right\} \quad (6{\cdot}15)$$

(ii) *Numerical solution*

Exactly as for the stress equations, the characteristic relations (6·12) are approximated by finite-difference relations which are then used to determine U and W at R (see figure 16), given their values at P and Q. There is one major difference between the determination of the velocity field and that of the stress field since the co-ordinates, and hence values of ψ, are already known at all three points P, Q and R. This situation is, of course, a direct reflexion of the fact that plastic régime F is statically determinate.

First approximations U_1 and W_1 to the values of U_R and W_R are determined from

$$\left.\begin{aligned}(U_1-U_P)\cos\phi &= (\psi_R-\psi_P)\,(W_P+U_P\sin\phi)\\ &\quad -\frac{1-\sin\phi}{r_R+r_P}\{U_P(r_R-r_P)\cos\phi-(W_P+U_P\sin\phi)\,(z_R-z_P)\},\\ (W_1-W_Q)\cos\phi &= -(\psi_R-\psi_Q)\,(U_Q+W_Q\sin\phi)\\ &\quad -\frac{1-\sin\phi}{r_R+r_Q}\{W_Q(r_R-r_Q)\cos\phi+(U_Q+W_Q\sin\phi)\,(z_R-z_Q)\}.\end{aligned}\right\}\qquad(6{\cdot}16)$$

Second approximations (here taken as final ones) to U_R and W_R are then found from (6·16) with $\frac{1}{2}(U_1+U_P)$ replacing U_P, $\frac{1}{2}(W_1+W_P)$ replacing W_P and so on, on the right-hand sides of these equations. Last, it must be verified that U_R and W_R satisfy the conditions (6·15) (here taken, of course, in finite-difference form) in order to complete the necessary checks on the validity of the solution.

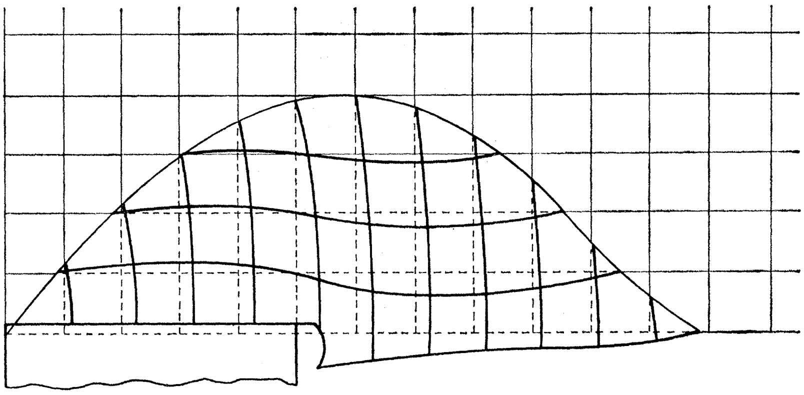

FIGURE 23. Resulting deformation of a square grid if the incipient velocity field is maintained for a short period of time for $\phi = 20°$.

The calculations outlined above were made on a desk calculating machine for the case $\phi = 20°$. For convenience, it is assumed that the solution near the origin is described to sufficient accuracy by that derived analytically in § 5 (*b*) as one mode for the incipient plastic deformation of a circular cylinder under uni-axial compression. The resulting velocity field, as reflected by the form of the incipient deformation undergone by a square grid, is depicted in figure 23. The conditions (6·15), checked at about one-in-ten of the mesh points, were found to be satisfied.

The velocity field has only been constructed for $\phi = 20°$ (and, by Shield 1955*b*, for $\phi = 0$). However, there is no reason *a priori* to suppose that it cannot be similarly constructed for all the values of ϕ of interest here. Assuming this to be true, the theorems of limit analysis then show that the values of the yield-point pressures p_3 obtained in § 6 (*a*) are upper bounds to the correct values.

(*c*) *Complete stress field*

The stress and velocity fields obtained in §§ 6 (*a*), (*b*) are, in Bishop's (1953) terminology, an *incomplete* solution, leading only to the conclusion that the yield-point loads obtained are upper bounds to the correct values. It is now necessary to prove that they are also lower bounds and hence are the correct values. Thus it must be shown that the stress field can be extended throughout the entire rigid region, without violating the conditions of equilibrium and yield. Only in this way is a *complete* solution derived.

The method used by Shield (1955*b*) for the case $\phi = 0$, involving the construction of a stress-free surface originating at the point *B*, is based upon that used by Bishop (1953) in the proof of two theorems which state whether or not such an extension is possible. However, in any particular case, if it is assumed that the required extension is possible in principle, then simpler fields can be constructed. In the present problem, the fact that an extension is possible for $\phi = 0$ suggests that the same is true for at least some range of positive values of ϕ; this conjecture is established directly through the construction of an acceptable extension.

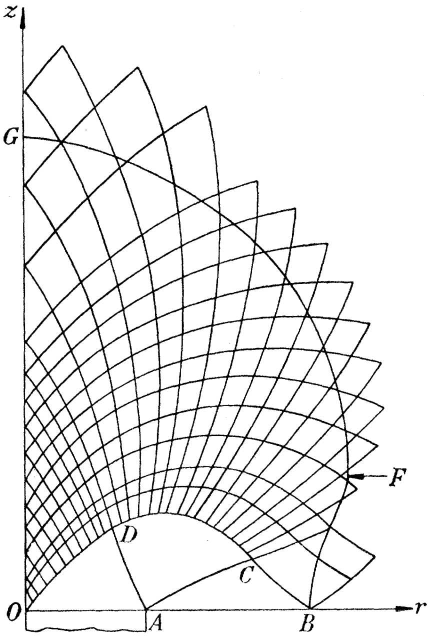

FIGURE 24. Extension of the stress field near the cylinder into the rigid region for $\phi = 20°$.

The method to be adopted follows that proposed by Bishop (1953) as one possible way of determining an acceptable extension of the stress field in the problem of the two-dimensional punch. Briefly, this method is as follows. Initially it is assumed the entire region exterior to *ODCB* is plastic, and the stress field is calculated in as much of this region as is necessary for the purposes of the following discussion. A curve *BFG* (see figure 24) is constructed in this region, starting at the known point *B* and ending at an initially unknown point *G* on the *z* axis, which is then regarded as a line of stress discontinuity. Across such a line of discontinuity, conditions of equilibrium require only that the normal and shear stresses be continuous, and therefore a certain amount of freedom still remains in the stress conditions exterior to *BFG*. The curve *BFG* is, in fact, so chosen that of these possible stress configurations one can be extended into the rest of the region in a simple way, described below, without violating the conditions of equilibrium and yield.

In more detail, the procedure is as follows. Both Λ and ψ are known on *ODCB* from the incomplete stress field derived in § 6(*a*). In addition, since the *z* axis is an axis of symmetry, which implies $\tau_{rz} = 0$ there, $\psi = \frac{3}{4}\pi - \frac{1}{2}\phi$ on $r = 0$. Thus, assuming that the behaviour of the medium is in accord with plastic régime *F*, sufficient boundary conditions are available to determine Λ and ψ in the region bounded by *ODCB*, the α line through *B* and the *z* axis. If it were necessary to calculate more of the field, the boundary conditions (6·6) are available on the *r* axis external to *OB*.

The numerical determination of this field, which is a *third boundary-value problem* (cf. Hill 1950), follows previous lines. The only difference now is that successive β lines are calculated clockwise from the *z* axis instead of counter-clockwise from the *r* axis.

Once this field is known the curve *BFG* is constructed as follows. It is convenient for the purposes of description to assume that the part *BF* of this curve has been calculated. The actual determination of *BF* is discussed later, but it may be noted that $\psi = \frac{1}{4}\pi - \frac{1}{2}\phi$ at *F*. The curve *FG* is then taken as the principal stress trajectory through *F* with initial direction parallel to the *z* axis. Bearing in mind the conditions that hold along a principal stress trajectory, it is easily seen that an acceptable stress configuration in the immediate exterior of *FG* is such that at any point there is only one non-zero component, say σ_n when referred to local Cartesian co-ordinates *n*, *s* respectively normal and parallel to *FG*. This stress configuration is then continued into the whole of the region lying above a line through *F* parallel to the *r* axis as follows. Let the curve *FG* be divided into a series of consecutive small elements. On any such element the force transmitted at each point is approximately constant and parallel to the normal to the element. This force can thus be held in equilibrium by a column, of constant cross-section, equal to that of the element, normal to the element and under a constant uni-axial stress equal to σ_n. Provided first that this stress σ_n, in absolute value, is less than or equal to the uni-axial yield stress $2c \tan(\frac{1}{4}\pi + \frac{1}{2}\phi)$ and secondly that the curve *FG* is everywhere concave to the punch, so that the columns do not intersect, the stress system comprising these stressed columns together with unstressed material between them gives, as a first approximation, an acceptable continuation of the stress field above a line through *F* parallel to the *r* axis. In the limit as the size of the elements tends to zero an acceptable stress field results.

The reason for considering the portion *BF* of the curve *BFG* separately is now apparent. In the region bounded by the *r* axis and the α line *AC* (see figure 24) the β lines are convex to the punch, so that any principal stress trajectory, which intersects the β lines at a constant angle, must also be convex. However, this difficulty is overcome by an extension of the above ideas, as follows. The curve *BF*, whose tangent at any point makes an angle ω with the *r* axis, is constructed in such a way that an acceptable stress field outside *BF* is one in which the only non-zero stress component is σ_r. The two equilibrium equations are sufficient to determine ω and σ_r, and provided σ_r, in absolute value, is less than or equal to $2c \tan(\frac{1}{4}\pi + \frac{1}{2}\phi)$ the stress field can again be extended using compression columns which are now, however, normal to the *z* axis. The curve *BF* is continued until, at the point *F*, ω again becomes equal to $\frac{1}{2}\pi$. In order that the principal stress trajectory *FG* may satisfy the requirements of the problem, *F* must lie above the α line *AC*. In the case considered below, this was so, but if it were not so the curve *BF* could be continued until this requirement was met.

The extension of the stress field discussed above has been calculated for $\phi = 20°$, and the resulting line of stress discontinuity *BFG* is shown in figure 24, together with the characteristic net; in the numerical work more characteristics were used than are shown in the figure. The stresses transmitted across this discontinuity were found to satisfy the conditions stated above.

In view of the fact that the required extension of the stress field has been shown to be possible for $\phi = 20°$, and, by Shield (1955*b*), for $\phi = 0$, it seems unlikely, since there are no new physical complications, that a complete solution cannot be found quite generally for $0° \leqslant \phi \leqslant 40°$. It is concluded that the yield-point loads obtained in § 6 (*a*) are also lower bounds and hence are equal to the correct values. In addition, it can be concluded that the stress fields that obtain in *OABCD*, found from the method discussed in § 6 (*a*), are the correct ones.

7. Concluding remarks

The present study of axially symmetric plastic deformations in ideal soils (with ductile metals, too, being included as a special case) has shown that real families of characteristics of the stress and velocity equations occur in all non-trivial cases. Now inasmuch as the solution of plasticity problems must involve attention, more often than not, to stress and velocity fields obtaining throughout regions with initially unknown boundaries, this striking situation makes for much simplicity in the mathematical and associated computational investigations. In particular, attention has been focused on the Haar & von Kármán type of plastic régimes as being, seemingly, of very fundamental significance in relation to the solution of certain classes of problems of interest. For these particular plastic régimes, the stress and velocity fields are hyperbolic with coincident families of characteristics, and the stress field is statically determinate under appropriate boundary conditions. In the present investigation, attention has been directed primarily towards the study of the Haar & von Kármán plastic régimes and their application to certain problems which not only serve as useful illustrations for the general mathematical theory but are also of physical interest in relation to soil mechanical testing and foundation engineering. In the problem of the indentation of a plane surface by a circular cylinder, it has been found that the yield-point loads increase markedly with the angle of internal friction. Although the details of the solution of the problem vary with this angle, the basic features remain unaltered. This fact suggests that the somewhat tedious construction of *complete* solutions (as distinct from *incomplete* ones) may well be omitted in proceeding to solve problems of a similar type. However, if such a procedure is followed, then careful judgement is obviously necessary. The present applications of the theoretical analysis considered in this paper are of course simpler than the analogous one involving effects due to soil weight or ones that involve finite displacements at the free surface.

Note added in proof (8 *June* 1961). The attention of the reader is drawn to Gvozdev's (1938, 1960) early work on the theorems of limit analysis and to Hill's (1961) recent work on the discontinuity relations in the mechanics of solids.

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